Included with the function in the integrand may be found a weight function $w(x)$. Various choices for the interval and the weight function permit developments of otherwise diverse functions under this single definition of orthogonality, such as the sine cosine Fourier series, the Legendre polynomials or surface zonal harmonics, Bessel functions, the polynomials of Tschebyscheff, Jacobi, Hermite and Laguerre. On the surface of a unit sphere, the tesseral harmonic functions, and the surface spherical harmonic functions of the same degree constitute orthogonal systems. Orthogonal functions provide a convenient method for the development of series as an approximation to an arbitrary function, after which conditions for convergence must be studied, as for example the Dirichlet conditions in Fourier's series. Approximations to a given function by a convergent series formed from orthogonal functions are valid over the entire interval, and as more terms are taken, the closeness of approximation is obtained simultaneously over the entire interval. This differs from Taylor's series, in which case the closeness of approximation is increased in the neighborhood of a point as more terms of the series are used.

2. In his "Recreation in Mathematics" Lick suggests the proposition: "Given any triangle. On its sides as bases construct isosceles triangles with equal thirty degree angles. The triangle formed by joining the vertices of these isosceles triangles is equilateral." Two such triangles are formed which are covariant with the given triangle. This paper discusses the properties of these covariant triangles as the given triangle is allowed to vary according to some fixed law.

3. A brief outline of the derivation of the Lagrangian equations of motion, of points moving under a conservative system of forces, is followed by a few applications: some of which demonstrate the method for determining the tensions and pressures due to certain geometrical or physical constraints.

4. Mr. Dunnington traced the development of the "prince of mathematicians" up to the attainment of his doctorate in 1799. Special attention was called to the textbooks by lesser known authors, and also to works of Euler, Newton, Lagrange and others, which he studied. His own influence on later mathematicians was pointed out and most of his later achievements were shown to have their origin in this period of his life. The more intimate and human side of Gauss during these early years was described; his relations with close friends, especially Wolfgang Bolyai, were discussed. Gauss rose to world fame from the very humblest origin. His father never had the faintest idea of his importance, but his mother lived to see her only child the doyen of German science. In 1792 he was considering the foundations of geometry; in 1794, discovery of the method of least squares; then for several years the theory of numbers, elliptic functions, series, the theory of functions, proof of the fundamental theorem of algebra, construction of the regular polygon of seventeen sides, and allied topics were in the foreground. The speaker has examined all the original sources on Gauss—in Göttingen, Brunswick, and elsewhere. Through friendship with the Gauss family he has access to a large mass of unpublished material and for several years has been at work on a full biography of him. An exhibit of

Gaussiana, including many of his original letters and rare pictures, was shown after the address.

LUCY T. DOUGHERTY, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirteenth regular meeting of the Southern California Section was held at Pomona College, Claremont, on Saturday, March 4, 1933. Professor L. D. Ames presided.

The attendance was fifty including the following thirty-one members of the Association: E. E. Allen, L. D. Ames, Harry Bateman, Clifford Bell, Grace E. Berry, Jessie R. Campbell, Myrtie Collier, P. H. Daus, Iva B. Ernsberger, Raymond Garver, H. H. Gaver, Harriet Glazier, J. M. Gleason, E. R. Hedrick, G. H. Hunt, C. G. Jaeger, G. A. Linhart, G. R. Livingston, W. E. Mason, A. D. Michal, G. F. McEwen, Lena E. Reynolds, G. E. F. Sherwood, Marcus Skarstedt, F. C. Touton, S. E. Urner, H. C. Van Buskirk, L. E. Wear, Mabel G. Whiting, H. C. Willett, E. R. Worthington.

The meeting began with a luncheon at the Claremont Inn, after which it adjourned to Mason Hall for a short business meeting and the program. The following officers were elected for the year 1933–34: Chairman, O. W. Albert, University of Redlands; Vice-Chairman, C. G. Jaeger, Pomona College; Program Committee, G. A. Linhart, Riverside Junior College and L. E. Wear, California Institute of Technology. The next meeting was tentatively scheduled for March 3, 1934 at Riverside Junior College.

A resolution of sympathy for the death of President Victor L. Duke of the University of Redlands was adopted and later transmitted to the proper persons.

The following program was presented:

1. "Time series: their analysis by successive smoothings" by Professor Lewis A. Maverick, Department of Economics, University of California at Los Angeles, by invitation.

2. "Linear functional differential equations" by Professor Harry Bateman, California Institute of Technology.

3. "The transformation $y=f'(x)$" by Professor Raymond Garver, University of California at Los Angeles.

4. "Estimation of vertical circulation and turbulence in the ocean from a solution of the equation of heat conduction" by Professor George F. McEwen, The Scripps Institute of Oceanography of the University of California.

5. "A formula for the profile of a stream" by Professor Edward Taylor, Pomona College, by invitation.

Abstracts of these papers follow, the numbers corresponding to the numbers of the titles.

1. It has been suggested by Ragnar Frisch that a trend or smoothing line may be drawn to pass through the points of inflection in the original curve, and that, if the smoothing line be itself examined, it will in turn show fluctuations

which will necessarily be marked by points of inflection. These points may be connected to give a trend or smoothing line of the second order, and it may be submitted to the same process. The first smoothing line is freed from the fluctuation of shortest period, but contains all fluctuations of longer period; the second smoothing line is in turn freed from the lowest order fluctuation remaining in the smoothing line of first order.

In the present paper the concept is accepted of a succession of smoothing lines which set apart the fluctuations into homogeneous orders. The criterion of points of inflection is accepted as but one of several criteria, aiding in the location of the successive smoothing lines; the short-coming lies in the indeterminateness of the location of the points of inflection. Added criteria employed are moving averages, minimum radius of curvature, and *bounds*, which are two curves that touch, respectively, the successive peaks and the successive troughs.

Each order fluctuation may be studied for period and amplitude and a standard pattern may be determined.

In forecasting, the highest order trend may be continued forward and those of lower orders superimposed in the form of their standard patterns. In correlation, the study may be carried on separately for each order fluctuation. In constructing index numbers, the selection and weighting of the component series may be undertaken separately for each order fluctuation, according to the significance of the series for the fluctuation of the particular order.

This paper will be published in full in the forthcoming issue of *Econometrica*.

2. A study of equations which are linear in a function of x, the same function of $-x$ and the derivatives of these functions. An operational calculus is developed with the operation of differentiation and the operation changing x into $-x$ as basic operations. Associated variational principles are considered.

3. The application of the transformation $y=f'(x)$ to the equation $f(x)=x^n+c_1x^{n-1}+\cdots+c_n=0$ leads to a transformed equation in y whose constant term, except possibly for sign, is the discriminant of $f(x)=0$. In the case of the cubic, quartic, and some special higher degree equations, this gives a convenient method of evaluating the discriminant. In fact, applied to the general nth degree equation, it leads to a determinant form for the discriminant which has certain advantages over the usual determinant forms.

4. The classical Fourier differential equation of heat conduction in which a term is included to take account of the circulation of the water is solved with special reference to conditions in the ocean. The diffusivity coefficient under these conditions is not a physical constant of the fluid but is a measure of the turbulence. The equation is solved so as to correspond to the observed initial vertical distribution of temperature below a level near the surface and the observed relation of temperature to time at that level. Tables have been prepared to facilitate the computation, and estimates are presented of turbulence and vertical flow based upon records of ocean temperatures in the San Diego region.

5. The formula for the long profile of a stream has been sought since 1875. Sternberg developed an exponential curve. Unwin developed a parabola. They

were both based upon such restriction and exception as to make their applications to the entire length of a stream impossible. The present analysis is based upon the hydraulic formulae of Chezy and Bazin. It shows that no single formula can represent all streams, nor all of a single stream; and that an analytic formula of the form y equals a single term in x is not possible.

P. H. DAUS, *Secretary*

THE TENTH ANNUAL MEETING OF THE MICHIGAN SECTION

The tenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan on Saturday, March 18, 1933, in conjunction with the Michigan Academy of Science. The chairman of the Section, Professor R. W. Clack, presided.

The attendance was over a hundred, including the following forty-four members of the Association: N. H. Anning, W. L. Ayres, W. D. Baten, Harold Blair, W. M. Borgman, J. W. Bradshaw, J. B. Brandeberry, R. W. Clack, C. J. Coe, J. J. Corliss, C. C. Craig, S. E. Crowe, Wayne Dancer, J. D. Elder, L. C. Emmons, J. P. Everett, Peter Field, W. B. Ford, B. C. Getchell, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, H. S. Kaltenborn, L. C. Karpinski, Theodore Lindquist, C. E. Love, A. L. Nelson, J. A. Nyswander, H. L. Olson, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, R. C. Shellenbarger, E. R. Sleight, A. G. Swanson, Dorothy Van Deusen, T. O. Walton, R. L. Wilder, J. B. Winslow.

A luncheon was held at noon at the Michigan Union. A short business session was held at this time and the following officers were chosen for the year 1933–34: Chairman, Professor T. O. Walton, Kalamazoo College; Secretary-Treasurer, Professor W. L. Ayres, University of Michigan; Third member of Executive Committee, Professor R. W. Clack, Alma College.

The following papers were read during the morning and afternoon sessions:

1. "The ellipsoidal viscosity distribution function for molecular velocities" by Professor W. S. Kimball, Michigan State College.

2. "Laplace transforms of conjugate nets" by W. M. Borgman, Jr., College of the City of Detroit.

3. "Differentiation with respect to a function" by Dr. Ben Dushnik, University of Michigan.

4. "The remainder theorem as a test for divisibility" by R. E. Gaskell, student, Albion College.

5. "Highlights of the Zürich Mathematical Congress" by Professor G. Y. Rainich, University of Michigan.

6. "Rare old books in mathematics" by Professor Harold Blair, Western State Teachers College.

7. "Double divisibility in a division algebra" by Professor H. L. Olson, Michigan State College.

8. "Geometrical studies suggested by a proof of the fundamental theorem of algebra" by Professor T. O. Walton, Kalamazoo College.

9. "An experiment in second semester mathematics for freshmen" by E. E. Ingalls, Albion College.

10. "Maya mathematics" by Director Carl Guthe, Museum of Anthropology, University of Michigan, by invitation.

11. "Practical harmonic analysis" by Professor N. H. Anning, University of Michigan.

12. "The introduction of the calculus into high school mathematics" by Professor R. F. McDaid, Michigan State Normal College.

Abstracts of the papers follow:

1. The paper discusses the change in the Maxwell spherical distribution function produced by a shearing force. It is shown that the distribution then becomes elliptical and that it satisfies Bolzmann's equation.

2. Nets of curves $u=c$ and $v=k$ on surfaces such that the tangents to the u-curves on one surface are tangent to the v-curves on the next surface are discussed. It is shown that if the tangents form a linear family then the surfaces are quadratic.

3. The derivative of a function $f(x)$ with respect to a function $\alpha(x, y)$ is defined as

$$f'_{\alpha}(x) = \lim_{y \to 0} \frac{f\{\alpha(x, y)\} - f\{\alpha(x, 0)\}}{\alpha(0, y)}$$

Dr. Dushnik proves that this operation has most of the elementary properties of ordinary derivatives except for some variation in the formulas for the derivative of a product and quotient. An illustrative example indicates a possible application of this operation to the solution of a certain type of functional operation.

4. The remainder theorem is used to develop the well-known tests for divisibility of numbers by 3, 9, and 11. Divisibility tests in number systems with a base other than 10 are also discussed and the results generalized.

5. The nature of Professor Rainich's paper is clear from the title.

6. Professor Blair exhibited and discussed some rare scientific books from the collection of the late Mr. A. M. Todd of Kalamazoo. These books are now in the library of the Western State Teachers College.

7. In this paper is developed a necessary and sufficient condition on an integral element v of a division algebra in order that, for every integral element x, there shall exist an integral element y such that $xv=vy$.

8. The paper studies some relations between the roots of the complex function $f(z)=u(x, y)+iv(x, y)$ and the curves $u=0$ and $v=0$. If $f(z)$ has a multiple root of order n at a point, then $u(x, y)=0$ and $v(x, y)=0$ will each have a multiple point of order n at this point. Further the branches of $u=0$ (and $v=0$) intersect at this point at angles of π/n, and the branches of $u=0$ bisect the angles between adjacent $v=0$ branches.

9. Three years ago the experiment of replacing trigonometry by special topics in charting and graphing, statistics, and compound interest was tried on a group of 40 students not planning to take further work in mathematics. Recently a questionnaire was sent to these students asking their opinions on the desirability of the course and whether they had actually used the knowledge gained from it. The paper discussed the content of the course and the favorable response to the questionnaire.

10. Dr. Guthe discussed the mathematics of the Mayas, "The Greeks of the New World." He gave a full account of their number system of base 20, showing their number symbols and their methods of combining them to represent large numbers. He described their calendar and pointed out that it was as accurate as and simpler than our Gregorian calendar. As there are no fractions in the Maya number system, he showed how, using integers only, they adjusted their man-made calendar to fit observed astronomical phenomena.

11. The paper was a summary of current methods—graphical, mechanical, and numerical. A Henrici-Coradi analyser, the property of the University of Michigan, was exhibited. This and the Mader machine were described in some detail. Professor Anning pointed out the advantages for rapid analysis of Runge's numerical method as modified by Zipperer and, more recently, by Terebesi.

12. The paper was an account of some lessons on maxima and minima given in the University High School, Ann Arbor, in October 1932.

W. L. AYRES, *Secretary*

ON THE NUMERICAL INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

By W. E. MILNE, Oregon State Agricultural College

Differential equations of the second order of the type

$$\frac{d^2y_i}{dx^2} = u_i(y_1, y_2, \cdots, y_n, x) \qquad (i = 1, 2, \cdots, n) \tag{1}$$

in which the first derivatives do not appear are of such frequent occurrence in physics, dynamics, and astronomy, that special methods for their numerical integration are of considerable interest and importance.*

The method here described is particularly well adapted for use with a calculating machine, provides a check on each step, and requires a minimum of written calculation. Trial computations using several methods indicate that when a calculating machine is available the method given below is the most rapid of all, though it is probable that some of the methods using differences are more satisfactory when all of the work has to be done by hand.

* A survey of various special methods of solving equations of type (1) numerically is given by Nyström, Acta Societatis Scientiarium Fennicae, vol. 50, page 14, #4A, page 23, #5A, pages 35–37.

1. For the sake of simplicity the treatment will be limited to a single differential equation with a single dependent variable, which may be written

$$d^2y/dx^2 = u(x, y). \tag{1}$$

The problem is to determine approximately the particular solution of this differential equation for which $y = y_0$, $dy/dx = y_0'$, when $x = x_0$. Let us assume that we have already obtained the values of y and of u corresponding to a set of uniformly spaced values $x_0, x_1, x_2, \cdots, x_n$, of the independent variable x, and that these quantities have been tabulated as follows:

$$\begin{array}{lll} x_0 & y_0 & u_0 \\ x_1 & y_1 & u_1 \\ x_2 & y_2 & u_2 \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ x_n & y_n & u_n \\ x_{n+1} & & \end{array}$$

The next step is to determine the values of y_{n+1} and u_{n+1} which are to be inserted opposite x_{n+1}. After these are found the next set of values for x_{n+2} will be secured by a precisely similar operation and thus the computation will proceed step by step.

For the calculation of y_{n+1} in each step we employ formulas of a special type which give y_{n+1} in terms of the tabulated y's and u's, and which do not contain the first derivative y'. The derivation of these special formulas is taken up in paragraph 2.

2. When the solution $y(x)$ is substituted for y in the expression $u(x, y)$ of equation (1) the function $u(x, y)$ becomes a function of x alone. We shall assume that this function u can be represented with a degree of accuracy[1] sufficient for our purpose by a few terms of Newton's interpolation formula, so that we may write

$$d^2y/dx^2 = u = u_n + \Delta u_n(x - x_n)/h + \Delta^2 u_n(x - x_n)(x - x_{n-1})/2h^2 + \cdots, \tag{2}$$

in which

$$\Delta u_n = u_n - u_{n-1},\ \Delta^2 u_n = \Delta u_n - \Delta u_{n-1},\ \text{etc.},\ h = x_n - x_{n-1}.$$

Integrating equation (2) twice and determining the constants so that $dy/dx = y_n'$, $y = y_n$, when $x = x_n$, we have

$$y(x) = y_n + y_n'(x - x_n) + h^2[u_n A_0(x) + \Delta u_n A_1(x) + \Delta^2 u_n A_2(x) + \cdots], \tag{3}$$

in which for the sake of brevity we have set

$$h^2 A_0(x) = (x - x_n)^2/2,$$

$$h^2 A_k(x) = \int_{x_n}^{x} ds \int_{x_n}^{s} (t - x_n)(t - x_{n-1}) \cdots (t - x_{n-k+1})\,dt/k!h^k,\ (k = 1, 2, \cdots).$$

[1] Cf., e.g., Steffensen, *Interpolation*, pp. 22–26.

The values of the first eight A's have been calculated for six different values of x and are given in the following table:

x	$A_0(x)$	$A_1(x)$	$A_2(x)$	$A_3(x)$	$A_4(x)$	$A_5(x)$	$A_6(x)$	$A_7(x)$
x_{n+1}	1	+1	3	+19	45	+863	9625	+67906
x_n	0	0	0	0	0	0	0	0
x_{n-1}	1	−1	−1	−4	−7	−107	−995	−6031
x_{n-2}	4	−8	0	−8	−16	−256	−2432	−14912
x_{n-3}	9	−27	27	−27	−27	−405	−3807	−23328
x_{n-4}	16	−64	128	−256	0	−512	−5120	−31744
x_{n-5}	25	−125	375	−1250	625	−1375	−6875	−40625
Common Denominator	2	6	24	180	480	10080	120960	907200

Equation (3) contains the first derivative y_n' and therefore is not directly applicable to our problem, but from equation (3) can be obtained a general formula of the desired type. Let m and k be any two integers, and substitute in (3) successively the following four values of x; $x=x_{m+1}$, $x=x_m$, $x=x_{k+1}$, $x=x_k$. Then we have four equations from which the term y_n' can be eliminated by subtracting the sum of the second and third from the sum of the first and fourth, so that we have the general equation

$$(4)\quad y_{m+1} - y_m - y_{k+1} + y_k = h^2 \sum_i \Delta^i u_n [A_i(x_{m+1}) - A_i(x_m) - A_i(x_{k+1}) + A_i(x_k)].$$

By giving m and k integral values in (4) we obtain a variety of particular formulas, all of which lack the term y_n'. Thus if $m=n$, $k=n-1$, we get, after transposition of the second, third, and fourth terms,

$$y_{n+1} = 2y_n - y_{n-1} + h^2[u_n + (1/12)\Delta^2 u_n + (1/12)\Delta^3 u_n + (19/240)\Delta^4 u_n + \cdots].$$

This equation is not new but was employed by Störmer.[1] We shall not use this particular case but shall endeavor to construct similar formulas in which the coefficient of the third difference shall vanish. It will be found that if $m=n$, $k=n-3$, we get

$$(5)\quad y_{n+1} = y_n + y_{n-2} - y_{n-3} + h^2[3u_n - 3\Delta u_n + (5/4)\Delta^2 u_n + (17/240)\Delta^4 u_n + \cdots].$$

while if $m=n-1$, $k=n-2$, we have

$$(6)\quad y_n = 2y_{n-1} - y_{n-2} + h^2[u_n - \Delta u_n + (1/12)\Delta^2 u_n - (1/240)\Delta^4 u_n + \cdots].$$

Now in (5) and (6) we substitute the identities

[1] Archives des Sciences physiques et naturelles, Genève, juillet–octobre 1907, p. 63ff.

$$\Delta u_n = u_n - u_{n-1}, \quad \Delta^2 u_n = u_n - 2u_{n-1} + u_{n-2},$$

neglect all terms containing differences of the fourth and higher orders, and obtain finally the two formulas that we actually use in calculation:

$$y_{n+1} = y_n + y_{n-2} - y_{n-3} + (h^2/4)(5u_n + 2u_{n-1} + 5u_{n-2}), \tag{7}$$

and

$$y_n = 2y_{n-1} - y_{n-2} + (h^2/12)(u_n + 10u_{n-1} + u_{n-2}). \tag{8}$$

It will be noted that if u is a polynomial of the third degree in x the fourth and higher differences vanish, so that the formulas (7) and (8) are exact. The exceptional simplicity of these two formulas is apparent when we observe that ordinarily a formula of this type which gives exact results for a polynomial of the third degree will contain the quantity u_{n-3} in addition to u_n, u_{n-1}, and u_{n-2}. Moreover we see from the table of A's that when the third difference does not cancel out the numerical coefficients will not be as simple as they are in (7) and (8). In their class these formulas are as noteworthy as Simpson's Rule is among quadrature formulas.

Equation (7) is of particular value for the integration of differential equations since it expresses y_{n+1} in terms of y_n, y_{n-1}, u_n, u_{n-1}, etc. thus permitting the evaluation of y_{n+1} by means of quantities already obtained. Equation (8) is used for recalculation, and in general is more accurate than (7).

Rigorous upper bounds for the magnitude of the error may be calculated in the usual manner, but as they involve the higher derivatives of $u(x, y)$ they are not usually convenient for practical use.

3. The actual procedure in integrating a differential equation is best shown by an example, for instance

$$\frac{d^2y}{dx^2} = -y^3,$$

with the initial values $y=0$, $dy/dx=1$, when $x=0$. Let $h=0.1$. In order to get started we need to have three values of y besides the initial value. These may be obtained from Taylor's series for y, a few terms of which are found to be

$$y = x - \frac{1}{20}x^5 + \frac{1}{480}x^9 + \cdots,$$

whence we obtain the following values

x	y	$u(x, y)$
−0.1	−0.0999995	0.0010000
0.0	0.0000000	0.0000000
0.1	0.0999995	−0.0010000
0.2	0.1999840	−0.0079981
0.3	. . . . .	. . . . .
0.4	. . . . .	. . . . .

To continue the computation we use (7) for $n=2$ to predict the value of y_3, obtaining $y_3=0.2998785$. From y_3 we find u_3 and now use (8) for $n=3$ to recalculate y_3, getting in this case precisely the same value as before. We conclude that y_3 is now correct, and proceed to predict y_4 by (7) and to recalculate y_4 by (8). The correction δy is $+1$ in the last figure, so y_4 and u_4 are corrected and we go on to y_5. The computation up to $x=0.7$ appears as follows when the corrected values are given. The column headed δy gives the correction.

x	y	u	δy
0.2	0.1999840	−0.007998	
0.3	0.2998785	−0.026967	
0.4	0.3994885	−0.063755	1
0.5	0.4984415	−0.123835	2
0.6	0.5961329	−0.211850	7
0.7	0.6916799	−0.330914	12

4. When fifth differences are negligible fourth differences are practically constant and we see by (6) that the error of (8) is $E_8=-h^2\Delta^4u_m/240$ and by (5) that the error of (7) is $E_7=17h^2\Delta^4u_n/240$ so that $\delta y=18h^2\Delta^4u_n/240$ and we have

$$E_8=\frac{\delta y}{18}, \tag{9}$$

Consequently from the tabulated values of δy we can form an estimate[1] of the error that is liable to occur in using formula (8). When the error given by (9) is large enough to affect the last digit retained it is advisable either to shorten the interval or to use more accurate formulas, such as those derived in the next section.

5. When in formula (4) we set $m=n$ and $k=n-5$, and afterwards replace the first four differences on the right by their values in terms of the u's we obtain a formula analogous to (7), which reads

$$y_{n+1}=y_n+y_{n-4}-y_{n-5}+\frac{h^2}{48}[67u_n-8u_{n-1}+122u_{n-2}-8u_{n-3}+67u_{n-4}]. \tag{10}$$

Again setting $m=n-1$, $k=n-4$ we obtain similarly a formula analogous to (8)

$$y_n=y_{n-1}+y_{n-3}-y_{n-4}+\frac{h^2}{240}[17u_n+232u_{n-1}+222u_{n-2}+232u_{n-3}+17u_{n-4}]. \tag{11}$$

These formulas are both exact if sixth differences of u vanish, and hence are in general considerably more accurate than (7) and (8).

[1] Of course (9) is strictly true only when fifth differences of u vanish, but it is nevertheless a very useful guide in actual practice.

6. Returning to our example we note that δy is increasing at such a rate as to make the error in the next calculation significant. We therefore abandon (7) and (8) and use instead (10) to predict, and (11) to correct, the values of y. The next five lines of the calculation appear as follows:

x	y	u	δy
0.8	0.7838909	−0.481689	+2
0.9	0.8712609	−0.661370	+1
1.0	0.9519989	−0.862799	−3
1.1	1.0241004	−1.074058	−5
1.2	1.0854666	−1.278938	−6

When seventh differences are negligible the error of (11) is

$$E_{11} = \frac{-318h^2}{120960}\Delta^6 u_n$$

and that of (10) is

$$E_{10} = \frac{7870h^2}{120960}\Delta^6 u_n$$

so that

$$\delta y = \frac{8188h^2}{120960}\Delta^6 u_n$$

and

$$E_{11} = -\frac{159}{4094}\delta y = -\frac{\delta y}{26} \text{ (approx.)} \tag{12}$$

When the error E_{11} given by (12) is significant it is advisable to choose a smaller value of h.

NEGATIVE-RECIPROCAL EQUATIONS

By H. S. UHLER, Yale University

In a certain investigation in geometrical optics I found it necessary to transform a rational algebraic equation in x to a new one in y by the relation $x = 2y(1-y^2)^{-1}$. $[x \equiv \tan 2\theta,\ y \equiv \tan \theta]$. The resulting equation in y was very closely related to the reciprocal equations discussed in the elementary treatises on the theory of equations, for example, in those by Todhunter, and by Burnside and Panton. The following examination of the chief properties of the less familiar type of equation was suggested by the original problem. The results may be of interest to others since they supplement and extend the discussions

given in the elementary text books.[1] The manner of presentation will parallel that of the texts both in style and in making no claim to formal rigor.

Definition: The epithet *negative-reciprocal* will be applied to a rational algebraic equation which (after multiplication by simple factors) is reproduced identically when its variable x is replaced by $-x^{-1}$.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0. \tag{1}$$

Substitute $-x^{-1}$ for x, and then multiply by x^n/p_n to obtain

$$x^n - \frac{p_{n-1}}{p_n}x^{n-1} + \frac{p_{n-2}}{p_n}x^{n-2} - \cdots \pm \frac{p_2}{p_n}x^2 \mp \frac{p_1}{p_n}x \pm \frac{1}{p_n} = 0, \tag{2}$$

where the upper or lower signs are to be taken according as n is even, or odd, respectively.

In order that equation (2) may be identical with equation (1) the following conditions must be fulfilled

$$p_1 = -\frac{p_{n-1}}{p_n}, \quad p_2 = \frac{p_{n-2}}{p_n}, \quad p_3 = -\frac{p_{n-3}}{p_n}, \cdots, p_{n-2} = \pm\frac{p_2}{p_n},$$

$$p_{n-1} = \mp\frac{p_1}{p_n}, \quad p_n = \pm\frac{1}{p_n};$$

the double signs conforming to the earlier qualification. The last equation gives $p_n^2 = \pm 1$ hence $p_n = \pm 1$ when n is even, and $p_n = \pm(-1)^{1/2} \equiv \pm i$ when n is odd. Accordingly, a negative-reciprocal equation of odd degree with exclusively real coefficients can not exist.

Case 1(a). n *even*, $p_n = +1$. Then

$$p_1 = -p_{n-1},\ p_2 = p_{n-2},\ p_3 = -p_{n-3}, \cdots, p_{n-3} = -p_3,$$

$$p_{n-2} = p_2,\ p_{n-1} = -p_1$$

or, more compactly,

$$p_j = (-1)^j p_{n-j};\ j = 1, 2, 3, \cdots, n-1.$$

Thus an equation is a negative-reciprocal equation when the coefficient of x^n and the constant term are both $+1$, and the coefficients of the terms at equal odd distances from the first and last are equal in magnitude and opposite in sign, while the coefficients of terms at equal even distances from the beginning and end agree both in magnitude and in sign. Furthermore, when $n/2$ is odd the middle term must vanish. $[n/2 = 2m-1 = j \ \therefore\ p_{2m-1} = (-1)^{2m-1}p_{2m-1} = -p_{2m-1}.]$ Then the self negative-reciprocal numbers $+i$ and $-i$ are roots of

[1] The broad features of the solution of the more general problem in which x and h/x are roots may be found in: J. Carnoy's *Cours d'Algèbre Supérieure*, pp. 256 to 262; and B. Niewenglowski's *Cours d'Algèbre*, vol. 2, pp. 344 to 348.

the equation, hence x^2+1 is a factor of the left member so that the degree of the equation can be reduced by two. The equation obtained by removing the factor in question is also a negative-reciprocal equation, and it likewise falls under the present case 1(a). Evidently $n/2$ is now even and hence further division by x^2+1 is, in general, no longer possible.

The total number of different but complete negative-reciprocal equations of degree n (n even, $p_n=+1$) in which the coefficients of the corresponding terms are equal in magnitude but have their signs permuted in every possible way consistent with the pairing required by the equations of condition, and the middle term (when present) is also alternately given the plus and the minus sign, is expressed by 2^e where e equals $n/2$ or $n/2-1$ according as $n/2$ is even or odd, respectively.

Case 1(b). *n even*, $p_n=-1$. Now

$$p_1 = p_{n-1},\ p_2 = -p_{n-2},\ p_3 = p_{n-3},\ \cdots,\ p_{n-3} = p_3,$$
$$p_{n-2} = -p_2,\ p_{n-1} = p_1$$

or

$$p_j = (-1)^{j+1}p_{n-j};\ j = 1, 2, 3, \cdots, n-1.$$

Hence, an equation is a negative-reciprocal equation when the coefficient of x^n and the constant term are $+1$ and -1 respectively, and the coefficients of the terms at equal even distances from the terms x^n and -1 are equal in absolute value but of contrary signs, while the coefficients of terms at equal odd distances from the extreme terms coincide both in magnitude and in sign. Moreover, when $n/2$ is even the term involving $x^{n/2}$ must be absent $[n/2=2m=j\ \therefore\ p_{2m}=(-1)^{2m+1}p_{2m}=-p_{2m}]$, and x^2+1 is a factor of the left member of the equation. Under these circumstances the degree of the polynomial can be lowered by two. The resulting equation is again a negative-reciprocal equation, and it too falls under the present case 1(b). Obviously $n/2$ is now odd so that further division by x^2+1 can not, in general, be performed.

The total number of distinct but complete negative-reciprocal equations of degree n (n even, $p_n=-1$) in which the coefficients of the corresponding terms are equal in absolute value but have their signs permuted in every possible arrangement compatible with the pairing necessitated by the conditional equations, while the middle term (when present) is also successively assigned the plus and the minus sign, is represented by 2^e where e equals $n/2-1$ or $n/2$ according as $n/2$ is even or odd, respectively.

Case 2(a). *n odd*, $p_n=+i$. Here

$$p_1 = ip_{n-1},\ p_2 = -ip_{n-2},\ p_3 = ip_{n-3},\ \cdots,\ p_{n-3} = -ip_3,$$
$$p_{n-2} = ip_2,\ p_{n-1} = -ip_1$$

or

$$p_j = (-1)^{j+1}ip_{n-j};\ j = 1, 2, 3, \cdots, n-1.$$

Accordingly, an equation is a negative-reciprocal equation when it has the following properties: the coefficient of x^n and the constant term are $+1$ and $+i$, respectively; the quotient obtained by dividing the coefficient of a term occupying an odd position after x^n by the coefficient of the term occupying the corresponding odd position before the constant term is $+i$; the quotient found by dividing the coefficient of a term occupying an even position following x^n by the coefficient of the term occupying the corresponding even position preceding the constant term is $-i$.

In the present case the equation has $+i$ or $-i$ as a root according as $(n+1)/2$ is even or odd, respectively. Correspondingly, the degree of the equation can always be depressed by unity to the next lower and even degree by removing either the factor $x-i$ or the factor $x+i$. The resulting equation is also always a negative-reciprocal equation having, in general, complex coefficients. For illustration

$$\begin{aligned} x^9 + a_1x^8 + a_2x^7 + a_3x^6 + a_4x^5 + ia_4x^4 - ia_3x^3 + ia_2x^2 - ia_1x + i \\ \equiv (x + i)\{x^8 + (a_1 - i)x^7 - [(1 - a_2) + ia_1]x^6 \\ - [(a_1 - a_3) - (1 - a_2)i]x^5 + [(1 - a_2 + a_4) + (a_1 - a_3)i]x^4 \\ + [(a_1 - a_3) - (1 - a_2)i]x^3 - [(1 - a_2) + ia_1]x^2 - (a_1 - i)x + 1\}. \end{aligned}$$

The special values $a_1=b_1+i$, $a_2=b_2+ib_1$, $a_3=b_3+ib_2$, and $a_4=b_4+ib_3$ are required to make all of the preceding coefficients real.

Case 2(b). *n odd*, $p_n=-i$. Then

$$p_1 = -ip_{n-1},\ p_2 = ip_{n-2},\ p_3 = -ip_{n-3},\ \cdots,\ p_{n-3} = ip_3,$$
$$p_{n-2} = -ip_2,\ p_{n-1} = ip_1$$

or

$$p_j = (-1)^j ip_{n-j};\ j = 1, 2, 3, \cdots, n-1.$$

Consequently, an equation is a negative-reciprocal equation when it possesses the following characteristics: the coefficient of x^n and the constant term are $+1$ and $-i$ respectively; the quotient obtained by dividing the coefficient of a term occupying an odd position after x^n by the coefficient of the term occupying the corresponding odd position before the constant term is $-i$; the quotient found by dividing the coefficient of a term occupying an even position following x^n by the coefficient of the term occupying the corresponding even position preceding the constant term is $+i$.

In the present case the equation has $+i$ or $-i$ as a root according as $(n+1)/2$ is odd or even, respectively. Correspondingly, the degree of the equation can always be decreased by one to the next lower and even degree by removing either the factor $x-i$ or the factor $x+i$. The resulting equation is also always a negative-reciprocal equation having, in general, complex coefficients. A typical example may be obtained by changing the signs before all the i's in the illustration with which case 2(a) closes.

The total number of different but complete negative-reciprocal equations of degree n (n odd, $p_n = +i$) in which the coefficients of the corresponding terms have equal moduli while the real units ± 1 and the pure imaginary i are independently permuted in every possible way consistent with the pairing required by the $(n-1)/2$ mutually independent equations of condition, is given by 2^{n-1}. Obviously this result also applies to the case in which $p_n = -i$ (n odd).

It is interesting to observe that the relation $x = \pm iy$ will transform any negative-reciprocal equation into an ordinary reciprocal equation (of the same degree), and vice versa. Let x_1 and x_2 be a pair of roots of a negative-reciprocal equation, then $x_1x_2 = -1$. Also define y_1 and y_2 by the relations $x_1 = \pm iy_1$ and $x_2 = \pm iy_2$, in which the signs at the same level must be taken together. By multiplication $x_1x_2 = i^2y_1y_2$, hence $y_1y_2 = +1$. That is to say, y_1 and y_2 constitute a pair of roots of an ordinary reciprocal equation. Obviously, if all the coefficients of the equation to be transformed are real and non-zero, certain ones of the coefficients of the transformed equation will contain the pure imaginary as an explicit factor. For illustration, the complete negative-reciprocal equation

$$x^8 + a_1x^7 + a_2x^6 + a_3x^5 + a_4x^4 - a_3x^3 + a_2x^2 - a_1x + 1 = 0$$

transforms into the two following ordinary reciprocal equations

$$y^8 \mp ia_1y^7 - a_2y^6 \pm ia_3y^5 + a_4y^4 \pm ia_3y^3 - a_2y^2 \mp ia_1y + 1 = 0$$

in which it may be supposed that none of the a's involves i as a factor.

Any negative-reciprocal equation of degree n can be reduced to the form

$$(3)\quad x^{2m} + c_1x^{2m-1} + c_2x^{2m-2} + \cdots + c_mx^m + \cdots + (-1)^{m-2}c_2x^2 + (-1)^{m-1}c_1x + (-1)^m = 0$$

by operating upon it as indicated by the fifth column of the following table.

n	p_n	$n/2$	$(n+1)/2$	Factor removed	m	Constant term
even	$+1$	even			$n/2$, even	$+1$
even	-1	odd			$n/2$, odd	-1
even	$+1$	odd		x^2+1	$(n/2)-1$, even	$+1$
even	-1	even		x^2+1	$(n/2)-1$, odd	-1
odd	$+i$		even	$x-i$	$(n-1)/2$, odd	-1
odd	$+i$		odd	$x+i$	$(n-1)/2$, even	$+1$
odd	$-i$		even	$x+i$	$(n-1)/2$, odd	-1
odd	$-i$		odd	$x-i$	$(n-1)/2$, even	$+1$

The coefficients (c) may be real or complex, and they are linear functions of the coefficients of the original, unreduced negative-reciprocal equation.

It will now be shown that the degree of equation (3) can always be depressed by one-half, that is, to m. By dividing this equation by x^m and pairing both the extreme terms and the terms which occupy positions at equal distances from x^{2m} and the constant term, it is found that

$$\left[x^m + \left(-\frac{1}{x}\right)^m\right] + c_1\left[x^{m-1} + \left(-\frac{1}{x}\right)^{m-1}\right]$$

$$+ c_2\left[x^{m-2} + \left(-\frac{1}{x}\right)^{m-2}\right] + \cdots \tag{4}$$

$$+ c_{m-2}\left[x^2 + \left(-\frac{1}{x}\right)^2\right] + c_{m-1}\left[x + \left(-\frac{1}{x}\right)\right] + c_m = 0.$$

Let

$$u \equiv x + (-1)(x^{-1}), \tag{5}$$

and

$$U_k \equiv x^k + (-1)^k(x^{-1})^k. \tag{6}$$

Then

$$U_k = u^k + ku^{k-2} + \frac{k(k-3)}{2!}u^{k-4} + \frac{k(k-4)(k-5)}{3!}u^{k-6}$$

$$+ \frac{k(k-5)(k-6)(k-7)}{4!}u^{k-8} + \cdots \tag{7}$$

$$+ \frac{k(k-r-1)(k-r-2)\cdots(k-2r+1)}{r!}u^{k-2r} + \cdots,$$

and

$$U_k = uU_{k-1} + U_{k-2}. \tag{8}$$

[See I. Todhunter's *Theory of Equations*, Art. 261 (1895).] Hence, definition (6) enables equation (4) to be written

$$U_m + c_1U_{m-1} + c_2U_{m-2} + \cdots + c_{m-2}U_2 + c_{m-1}U_1 + c_m = 0. \tag{9}$$

Formula (7) shows at once that equation (9) is of degree m in u, and this is tantamount to saying that the degree of equation (3) has been decreased to one-half its original value.

Let u_0 be a root of equation (9). Then definition (5) gives the quadratic

$$x^2 - u_0x - 1 = 0. \tag{10}$$

The constant term of this equation shows that the product of its two roots equals -1, that is to say, either root is the negative-reciprocal of the other. Thus equation (10) pairs the related roots of equation (3).

Each of the roots obtained by equating to zero the factors given in the last six rows of the fifth column of the preceding table, namely $+i$ and $-i$, is the negative reciprocal of itself.

The following list of values of U_k will facilitate the reduction of negative-reciprocal equations of degree at most as high as sixteen.

$$U_1 = u, \qquad U_5 = u^5 + 5u^3 + 5u,$$
$$U_2 = u^2 + 2, \qquad U_6 = u^6 + 6u^4 + 9u^2 + 2,$$
$$U_3 = u^3 + 3u, \qquad U_7 = u^7 + 7u^5 + 14u^3 + 7u,$$
$$U_4 = u^4 + 4u^2 + 2, \qquad U_8 = u^8 + 8u^6 + 20u^4 + 16u^2 + 2.$$

In conclusion attention may be called to the fact that an unlimited number of equations of transformation which will convert any rational algebraic equation into a negative-reciprocal equation of higher degree can be constructed in the following obvious manner. First write an irreducible negative-reciprocal equation having literal coefficients and y as the variable. Next replace all of the symbols for these coefficients by the single letter x, (or by $1/x$), and then solve for x. For illustration, put $a_1 = a_2 = x$ in

$$y^4 + a_1y^3 + a_2y^2 - a_1y + 1 = 0$$

to obtain

$$x = (1 + y^4)/[y(1 - y - y^2)].$$

ON THE CLASSIFICATION OF COLLINEATIONS IN THE PLANE

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1. *Introduction.* A collineation in the plane is a projective transformation of a plane of points into itself in such a way that points which are on a line are transformed into points which are on a line. Analytically, a collineation is defined by equations of the type

$$\rho x_i' = \sum a_{ij}x_j \qquad (i = 1, 2, 3), \tag{1}$$

where the a's are any numbers, real or complex, subject to the single restriction that the determinant, $\Delta = |a_{ij}|$, shall be different from zero. Here (x_1, x_2, x_3) are the homogeneous coordinates of a point $P(x)$ and (x_1', x_2', x_3') are the coordinates of the transformed point $P'(x')$.

If the a's are complex and if we interpret any triple of numbers, $(x_1, x_2, x_3) \neq (0, 0, 0)$, real or complex, as the coordinates of a point in a plane, there are, exclusive of the identity, five distinct types of collineations in the plane.[1] If, however, the collineation is *real*, i.e., the a's are real numbers, and if we confine our attention to *real* points (x_1, x_2, x_3), one of these types subdivides further, so that, exclusive of the identity, there are six types of real collineations in the plane.

[1] Cf. for example, the classifications in Veblen and Young, *Projective Geometry*, Vol. I, p. 106, p. 273; Winger, *Projective Geometry*, pp. 298–306; Woods, *Higher Geometry*, pp. 83–86.

This classification can be made from either the geometric or the algebraic standpoint.

Most of the American texts on analytic projective geometry adopt the geometric viewpoint and make the classification through the orientation of the fixed elements. Those, however, that do make an algebraic classification employ the theory of elementary divisors,[1] an elegant and powerful theory and one which is practically indispensable in case four or more variables are involved. But that theory is much more elaborate and complicated than is necessary for the problem under consideration. Moreover, the average student of projective geometry is totally unacquainted with it at that stage in his mathematical career. It is probably for the purpose of avoiding this extensive algebraic theory that many writers adopt the geometric viewpoint rather than the algebraic, the latter being, in the opinion of the writer, much the more satisfactory of the two.

In this paper we are interested primarily in an algebraic classification. However, no use is made of the theory of elementary divisors. On the contrary, the discussion presupposes a knowledge of only the simplest facts from the theory of equations with which every student of projective geometry should be familiar.

It is true that any instructor in projective geometry might work up for his class just such a classification as is made here, and doubtless many of them do so. Still, there should be some convenient place in the literature to which the instructor might refer his students before taking up the subject in class.

2. *The induced line transformation.* Suppose that the point x' runs along the line u' so that $\sum u_i' x_i' = 0$. We then have by (1)

$$\sum a_{ij} u_i' x_j = \rho \sum u_i' x_i' = 0$$

so that the corresponding point x runs along the line u, where

$$\sigma u_j = \sum a_{ij} u_i' \qquad (j = 1, 2, 3). \tag{2}$$

Hence, the point transformation (1) induces the line transformation (2).

3. *Fixed elements under a collineation.* A point x is transformed into itself by (1), or is a fixed point of the collineation, provided there exists a number $\rho \neq 0$, such that

$$\sum a_{ij} x_j = \rho x_i \qquad (i = 1, 2, 3),$$

or

$$\begin{aligned} (a_{11} - \rho)x_1 + a_{12}x_2 + a_{13}x_3 &= 0, \\ a_{21}x_1 + (a_{22} - \rho)x_2 + a_{23}x_3 &= 0, \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \rho)x_3 &= 0. \end{aligned} \tag{3}$$

[1] For example, both Veblen and Young, and Woods refer to Bôcher, *Introduction to Higher Algebra*, Chaps. XX and XXI, wherein is given an exposition of the theory of elementary divisors as a basis for an algebraic classification.

This system of equations will be satisfied by a set of values $(x_1, x_2, x_3) \neq (0, 0, 0)$ if, and only if, ρ be so chosen that the determinant

$$D(\rho) = \begin{vmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{21} & a_{22} - \rho & a_{23} \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix} \tag{4}$$

vanishes.

The matrix of the coefficients of the system of equations (3) is called the *characteristic matrix* of the collineation (1) and is denoted by $A - \rho I$. The determinant $D(\rho)$ of this matrix is the *characteristic determinant* and the equation $D(\rho) = 0$ is the *characteristic equation* of the collineation. When expanded, this equation is

$$D(\rho) \equiv -\rho^3 + \rho^2 \sum a_{ii} - \rho \sum A_{ii} + \Delta = 0, \tag{5}$$

where A_{ii} is the cofactor of a_{ii} in $\Delta = |a_{ij}|$.

Corresponding to each root ρ of (5) there exists at least one set of solutions of the system (3); i.e., at least one fixed point of the collineation.

Similarly, a line u will be a fixed line under the collineation provided there exists a number $\rho \neq 0$, such that

$$\sum a_{ji} u_j = \rho u_i \qquad (i = 1, 2, 3). \tag{6}$$

These equations will be satisfied by a set of u's, not all zero, if, and only if, ρ be so chosen that the determinant of the coefficients in (6) vanishes. This is seen to be precisely the condition $D(\rho) = 0$.

THEOREM I. *Corresponding to each root of the characteristic equation $D(\rho) = 0$, of the collineation* (1), *there is at least one fixed point and one fixed line of the collineation.*

It is clear that the matrix of the system (6) is precisely the transpose of the matrix of the system (3). Hence, the ranks of the two matrices are the same. Since the number of linearly independent solutions of a system of homogeneous linear equations in n unknowns depends entirely on the rank of the matrix of the system,[1] we have the theorem:

THEOREM II. *Corresponding to any root ρ of the equation $D(\rho) = 0$, there are just as many linearly independent fixed lines as fixed points of the collineation.*

THEOREM III. *The rank of the matrix $A - \rho I$ is never less than 2 unless ρ is a multiple root of $D(\rho) = 0$.*

For if the rank of this matrix is less than 2, all the second order principal minors vanish, so that in addition to $D(\rho) = 0$, we have also

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[1] Bôcher, *loc. cit.*, Theorem 1, p. 50.

$$\rho^2 - (a_{11} + a_{22})\rho + A_{33} = 0,$$
$$\rho^2 - (a_{11} + a_{33})\rho + A_{22} = 0,$$
$$\rho^2 - (a_{22} + a_{33})\rho + A_{11} = 0.$$

On adding we get

$$3\rho^2 - 2\rho \sum a_{ii} + \sum A_{ii} = 0,$$

or

$$D'(\rho) = 0.$$

Hence, ρ is a multiple root of $D(\rho) = 0$.

The rank of $A - \rho I$ will be zero, i.e., every element of the matrix will be zero, if and only if $a_{ij} = 0$, $i \neq j$, and $a_{11} - \rho = a_{22} - \rho = a_{33} - \rho = 0$, so that $a_{11} = a_{22} = a_{33}$. The transformation is then the identity:

$$\tau x_i' = k x_i \qquad (i = 1, 2, 3).$$

THEOREM IV. *The rank of the matrix $A - \rho I$ is never zero unless the collineation* (1) *reduces to the identity.*

Since the characteristic equation of the identical transformation has a triple root, we have the corollary:

COROLLARY. *The rank of $A - \rho I$ is never less than* 1 *unless the equation $D(\rho) = 0$ has a triple root.*

4. *Incidence of fixed points and fixed lines.*

THEOREM V. *The same fixed point (line) cannot arise from two distinct roots of $D(\rho) = 0$.*

For suppose that the point x arose from the two distinct roots ρ and σ, so that we would have as in (3):

$$\sum a_{ij} x_j = \rho x_i, \qquad \sum a_{ij} x_j = \sigma x_i \qquad (i = 1, 2, 3).$$

On subtracting these two equations, member for member, we have

$$(\rho - \sigma) x_i = 0 \qquad (i = 1, 2, 3),$$

whence, since $\rho \neq \sigma$, $(x_1, x_2, x_3) = (0, 0, 0)$ contrary to supposition.

THEOREM VI. *A fixed line u arising from one root ρ of $D(\rho) = 0$ always passes through a fixed point x arising from a root $\sigma \neq \rho$.*

For the coordinates of the fixed line satisfy the equations

$$\sum a_{ji} u_j = \rho u_i, \tag{6'}$$

and the coordinates of the fixed point satisfy the equations

$$\sum a_{ij} x_j = \sigma x_i. \tag{7}$$

Multiply (6′) through by x_i and sum as to i; similarly, multiply (7) through by u_i and sum as to i. We then obtain:

$$\sum a_{ji}u_jx_i = \rho \sum u_ix_i,$$
$$\sum a_{ij}u_ix_j = \sigma \sum u_ix_i.$$

Since the left hand members of these last two equations are identical, we have on subtracting

$$(\rho - \sigma) \sum u_ix_i = 0,$$

whence, since $\rho \neq \sigma$, $\sum u_ix_i = 0$.

THEOREM VII.[1] *If the matrix $A - \rho_1 I$ is of rank 2, the fixed point x and the fixed line u arising from the root ρ_1 will be incident if, and only if, ρ_1 is a multiple root of $D(\rho) = 0$.*

Let us choose the fixed point x arising from the root ρ_1 as the vertex (1, 0, 0) of our triangle or reference. It will then follow that

$$a_{11} = \rho_1, \; a_{21} = a_{31} = 0,$$

so that the matrix A is of the form

$$A = \begin{pmatrix} \rho_1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

The three characteristic roots of A are obviously ρ_1 and the two roots of

$$F(\rho) = \begin{vmatrix} a_{22} - \rho & a_{23} \\ a_{32} & a_{33} - \rho \end{vmatrix} = 0.$$

The coordinates of the fixed line u arising from the root ρ_1 may be taken as

$$u_1 : u_2 : u_3 = \begin{vmatrix} a_{22} - \rho_1 & a_{23} \\ a_{32} & a_{33} - \rho_1 \end{vmatrix} : \begin{vmatrix} a_{32} & a_{33} - \rho_1 \\ a_{12} & a_{13} \end{vmatrix} : \begin{vmatrix} a_{12} & a_{13} \\ a_{22} - \rho_1 & a_{23} \end{vmatrix}.$$

This line will pass through the point x (1, 0, 0) if, and only if, $F(\rho_1) = 0$, i.e., ρ_1 is a multiple root of $D(\rho) = 0$.

5. *The classification of collineations.* We are now in a position to classify collineations.

Type I. $D(\rho) = 0$ *has three distinct roots*, ρ_1, ρ_2, ρ_3.

By Theorem III, each matrix $A - \rho_i I$ is of rank 2, so that corresponding to each root ρ_i there is a single fixed point P_i and a single fixed line p_i of the collineation. Moreover, by Theorem VI, p_1 passes through P_2 and P_3, but, by

[1] My thanks are due to Professor J. W. Lasley, Jr. for a suggestion as to the proof of this theorem.

Theorem VII, does not pass through P_1; etc. The three fixed points are therefore the vertices of a proper triangle where p_i is the side opposite the vertex P_i. If we choose this triangle as the triangle of reference so that $x_1=0$, $x_2=0$ and $x_3=0$ are fixed lines, and (1, 0, 0), (0, 1, 0) and (0, 0, 1) are fixed points, it is easily shown that the equations of the collineation are[1]

$$\tau x_i' = \rho_i x_i \qquad (i = 1, 2, 3).$$

Next, suppose that the characteristic equation $D(\rho)=0$ has a double root ρ_1 and a simple root ρ_3. The matrix $A-\rho_3 I$ is of rank 2 by Theorem III. There is a separation of cases according as $A-\rho_1 I$ is of rank 2 or rank 1. We consider the latter type first.

Type II. $D(\rho)=0$ *has roots* ρ_1, ρ_1, ρ_3; $A-\rho_1 I$ *of rank* 1. Corresponding to the root ρ_1, there is an entire line p of fixed points and an entire pencil of fixed lines with vertex at P, say. Corresponding to the root ρ_3, there is a single fixed point Q and a single fixed line q, where, by Theorem VII, Q does not lie on q. Since by Theorem VI, q passes through every point arising from the root ρ_1, obviously $q \equiv p$. Since Q lies on every line arising from the root ρ_1, Q is the vertex of the pencil of these lines and $Q \equiv P$. The fixed elements therefore consist of a line p of fixed points and a pencil P of fixed lines, where P is not on p[2]. If we choose as the vertices of our triangle of reference the points $P(0, 0, 1)$ and two distinct points $R(1, 0, 0)$ and $S(0, 1, 0)$ on p, it is easily shown that the equations of the collineation are in the canonical form:[3]

$$\begin{aligned} \tau x_1' &= \rho_1 x_1 \\ \tau x_2' &= \rho_1 x_2 \\ \tau x_3' &= \rho_3 x_3. \end{aligned}$$

This is the so-called *homology*. If $\rho_3 = -\rho_1$, we have the harmonic homology or reflexion.

Type III. $D(\rho)=0$ *has roots* ρ_1, ρ_1, ρ_3; $A-\rho_1 I$ *of rank* 2. Corresponding to the root ρ_1 there is a single fixed point P and a single fixed line p, where, by Theorem VII, P lies on p. Corresponding to the root ρ_3 there is a single fixed point Q and a single fixed line q, where Q does not lie on q. But by Theorem VI, P lies on q and Q lies on p. Hence P is the point of intersection of p and q.[4]

If we choose as vertices of our triangle of reference the points Q (0, 0, 1), P (1, 0, 0) and R (0, 1, 0), some point on q distinct from P, the equations of the collineation are easily shown to be:

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[1] Winger, *loc. cit.*, p. 300.

[2] That $P=Q$ is not a point of p follows also from Theorem V.

[3] Winger, *loc. cit.*, pp. 301–2.

[4] The point of intersection of two fixed lines of the collineation is obviously a fixed point of the collineation. Hence, p and q intersect in either P or Q; but we cannot tell which of these points lies at the intersection of p and q, unless we go back to Type I and invoke the principle of continuity, or else use some theorem such as VII.

$$\begin{aligned} \tau x_1' &= \rho_1 x_1 + k x_2 \\ \tau x_2' &= \rho_1 x_2 \qquad (k \neq 0). \\ \tau x_3' &= \rho_3 x_3. \end{aligned}$$

We consider finally the cases where the characteristic equation $D(\rho) = 0$ has a triple root ρ, ρ, ρ. There is a separation of cases depending on the rank of $A - \rho I$, which may be 2, 1 or 0.

Type IV. $D(\rho) = 0$ *has a triple root* ρ; $A - \rho I$ *of rank* 2. In this case there is a single fixed point P and a single fixed line p, where, by Theorem VII, P lies on p. If we choose P as the vertex (0, 0, 1) of our triangle of reference and p as the side $x_1 = 0$, the equations of the collineation are easily shown to be of the form:

$$\begin{aligned} \tau x_1' &= \rho x_1 \\ \tau x_2' &= a_{21} x_1 + \rho x_2 \qquad (a_{21} a_{32} \neq 0). \\ \tau x_3' &= a_{31} x_1 + a_{32} x_2 + \rho x_3. \end{aligned}$$

Type V. $D(\rho) = 0$ *has a triple root* ρ; $A - \rho I$ *of rank* 1. Corresponding to the root ρ there is an entire line p of fixed points and an entire pencil P of fixed lines. Obviously, the vertex P of the pencil of fixed lines is a fixed point, so that P must be one of the points of p. This is the so-called *elation*. If we choose P as the vertex (0, 0, 1) and p as the side $x_1 = 0$ of our triangle of reference, the equations of the collineation reduce to the canonical form:

$$\begin{aligned} \tau x_1' &= \rho x_1 \\ \tau x_2' &= \rho x_2 \qquad (k \neq 0). \\ \tau x_3' &= k x_1 + \rho x_3. \end{aligned}$$

Type VI. $D(\rho) = 0$ *has a triple root* ρ; $A - \rho I$ *of rank* 0. By Theorem IV, this type of collineation is the identity.

6. *Real collineations.* The above classification has been made on the assumption that any triple of numbers, $(x_1, x_2, x_3) \neq (0, 0, 0)$, real or complex, are to be interpreted as the coordinates of a point in the plane. In case we confine our attention to the geometry of reals, where the a's in (1) are real and where triples of real numbers only are to be interpreted as the coordinates of a point, the classification is somewhat different.

By reference to (3) and (6) it is obvious that when the a's are real, a fixed point x (line u) will be real if, and only if, ρ is real. Hence, since a real cubic has a multiple root only when all its roots are real, Types II, $\cdots$, VI arise in the case of real collineations just as before, with the understanding that *fixed* point (line) means *real* fixed point (line). Type I, however, subdivides into two types depending on whether $D(\rho) = 0$ has *one* or *three* real roots.

Type I. $D(\rho) = 0$ *has three distinct real roots.* In this case we have Type I as listed above where fixed point (line) is understood to mean real fixed point (line).

Type I′. $D(\rho)=0$ *has one real root and two complex roots.* Corresponding to the real root ρ_1 there is a single real fixed point P_1 and a single real fixed line p_1, where, by Theorem VII, p_1 does not pass through P_1. If now we choose P_1 as the vertex (1, 0, 0) and the line p_1 as the side $x_1=0$ of our triangle of reference, the collineation assumes the canonical form

$$\begin{aligned} \tau x_1' &= \rho_1 x_1 \\ \tau x_2' &= a_{22}x_2 + a_{23}x_3 \\ \tau x_3' &= a_{32}x_2 + a_{33}x_3, \end{aligned}$$

where the quadratic

$$F(\rho) = \rho^2 - (a_{22} + a_{33})\rho + a_{22}a_{33} - a_{23}a_{32} = 0,$$

has imaginary roots, i.e.,

$$(a_{22} - a_{33})^2 + 4a_{23}a_{32} < 0.$$

A Correction

Professor Morgan Ward has kindly called my attention to an unintentional misstatement of Theorem I of my note "*On a Certain Transformation of Infinite Series*" in the April number of this Monthly (vol. 40, p. 226). It should read as follows:

If $\lim_{n\to\infty} nu_n = l$ *exists, then the two series* (U) *and* (V) *both diverge, if* $l\neq 0$. *If* $l=0$, *the convergence of one implies that of the other, and the two series have the same sum.*

J. A. Shohat

QUESTIONS, DISCUSSIONS, AND NOTES

Edited by R. E. Gilman, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A Certain Class of Trigonometric Integrals

By Morgan Ward, California Institute of Technology

1. In the December issue of the Monthly[1] Professor Uhler has raised some questions about the functions defined by the indefinite integrals

$$\int \begin{matrix}\cos\\ \sin\end{matrix}(\cot\theta) \begin{matrix}\cos\\ \sin\end{matrix}\theta\, d\theta$$

[1] American Mathematical Monthly, vol. 39 (1932), p. 589.

which I propose to answer here. Let us define four functions of the real variable θ:

$$K_1(\theta) = \int_0^\theta \cos(\tan\phi)\cos\phi d\phi, \quad K_2(\theta) = \int_0^\theta \cos(\tan\phi)\sin\phi d\phi,$$

$$K_3(\theta) = \int_0^\theta \sin(\tan\phi)\cos\phi d\phi, \quad K_4(\theta) = \int_0^\theta \sin(\tan\phi)\sin\phi d\phi.$$

The integrals under discussion are immediately expressible in terms of these functions; for example,

$$\int \cos(\cot\theta)\sin\theta d\theta = \text{const.} - K_1\left(\frac{\pi}{2} - \theta\right).$$

We shall show that the functions $K(\theta)$ are not expressible in finite terms by any simple known functions. However, in the range $-\pi/2 < \theta < \pi/2$, they are representable by convergent series of which

$$(1.1)\quad K_1(\theta) = \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{1}{6}\cos\frac{3\theta}{2}\left(2\sin\frac{\theta}{2}\right)^3 - \frac{1}{8}\sin 2\theta\left(2\sin\frac{\theta}{2}\right)^4$$
$$+ \frac{1}{30}\cos\frac{5\theta}{2}\left(2\sin\frac{\theta}{2}\right)^5 - \frac{1}{36}\sin 3\theta\left(2\sin\frac{\theta}{2}\right)^6$$
$$+ \frac{2}{45}\cos\frac{7\theta}{2}\left(2\sin\frac{\theta}{2}\right)^7 + \frac{1}{30}\sin 4\theta\left(2\sin\frac{\theta}{2}\right)^8 + \cdots$$

may be quoted as typical. I shall give recursion formulas by which the numerical coefficients in these series may be calculated, and an estimate of the error terms. It turns out that the convergence is fairly good in the range $-\pi/4 \leqq \theta \leqq \pi/4$.

There also exist asymptotic expansions giving the behavior of the functions near $\pi/2$ and $-\pi/2$ which limitations of space forbid my developing here.

2. We begin by observing that from our defining relations, it follows that[1]

$$K_1(\theta+\pi) = -K_1(\theta); \quad K_2(\theta+\pi) = 2K_2\left(\frac{\pi}{2}\right) - K_2(\theta);$$

$$K_3(\theta+\pi) = 2K_3\left(\frac{\pi}{2}\right) - K_3(\theta); \quad K_4(\theta+\pi) = -K_4(\theta).$$

We may therefore assume that $-\pi/2 \leqq \theta \leqq \pi/2$.

If we let

$$P(\theta) = K_1(\theta) + iK_2(\theta), \quad Q(\theta) = K_3(\theta) + iK_4(\theta),$$

we obtain immediately from (1.1) the integral formulas

[1] The constants $K_2(\pi/2)$, $K_3(\pi/2)$ may be expressed as infinite integrals by writing $\tan\phi = x$, and these integrals may be evaluated in terms of Bessel functions, and related expressions.

$$P(\theta) + iQ(\theta) = \int_0^\theta \cos(\tan\phi)e^{i\phi}d\phi + i\int_0^\theta \sin(\tan\phi)e^{i\phi}d\phi = \int_0^\theta e^{i(\tan\phi+\phi)}d\phi,$$

$$P(\theta) - iQ(\theta) = \int_0^\theta \cos(\tan\phi)e^{i\phi}d\phi - i\int_0^\theta \sin(\tan\phi)e^{i\phi}d\phi = \int_0^\theta e^{-i(\tan\phi-\phi)}d\phi.$$

We shall now introduce two functions of a complex variable in terms of which these last integrals are easily expressible.

3. Let Z denote a complex variable. Consider the two functions

$$(3.1) \qquad F(Z) = \int_1^Z \exp\left(\frac{1-z^2}{1+z^2}\right)dz, \quad G(Z) = \int_1^Z \exp\left(\frac{z^2-1}{z^2+1}\right)dz.$$

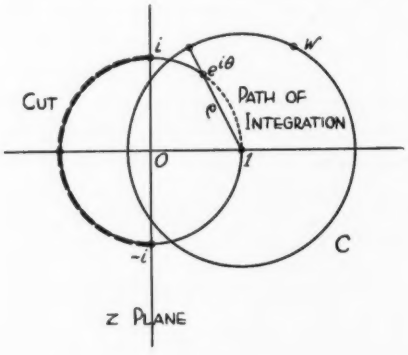
Then if we join the points $z=i$ and $z=-i$ by a cut which it is convenient to take along the left half of the unit circle (see figure), the reader may verify that

(a) The functions $F(Z)$ and $G(Z)$ are one-valued and analytic at all points Z of the cut z-plane save the point at infinity where each has a pole of order one, and their value at any point Z is independent of the path of integration joining 1 and Z.

(*b*) In particular, both functions are one-valued and analytic in the interior of a circle of radius $\sqrt{2}$ about the point $Z=1$.

(c) Both functions have essential singularities and logarithmic branch points at $z=i$ and $z=-i$, but remain finite as we approach these points along the right half of the unit circle.

(*d*) Upon a circle C with centre 1 and radius $\rho<\sqrt{2}$,



$$|F(Z)| \leqq \rho \exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)}, \quad |G(Z)| \leqq \rho \exp\frac{\sqrt{\rho^4+4}-\rho^2}{2(2-\rho^2)}.$$

4. Now let $Z=e^{i\theta}$, $-\pi/2<\theta<\pi/2$, be a point on the right half of the unit circle. Then if in the integrals (3.1) we choose for our path of integration the arc of the unit circle from 0 to θ, we obtain on writing $e^{i\phi}$ for z and reducing,

$$F(e^{i\theta}) = i\int_0^\theta e^{-i(\tan\phi-\phi)}d\phi, \quad G(e^{i\theta}) = i\int_0^\theta e^{i(\tan\phi+\phi)}d\phi.$$

On combining these results with the formulas of sections 1 and 2, we find that

$$\begin{aligned} K_1(\theta) &= \text{Real part of } \frac{F(e^{i\theta}) + G(e^{i\theta})}{2i}, \\ K_2(\theta) &= \text{Imaginary part of } \frac{F(e^{i\theta}) + G(e^{i\theta})}{2i}, \\ K_3(\theta) &= \text{Real part of } \frac{F(e^{i\theta}) - G(e^{i\theta})}{2}, \\ K_4(\theta) &= \text{Imaginary part of } \frac{F(e^{i\theta}) - G(e^{i\theta})}{2}. \end{aligned} \tag{4.1}$$

It is clear from these formulas that the function-theoretic nature of the $K(\theta)$ is determined by that of $F(Z)$ and $G(Z)$.

5. The nature of the functions $F(Z)$ and $G(Z)$ may be best seen from the differential equations which they satisfy. From formula (3.1)[1]

$$\frac{dF}{dz} = \exp\frac{1-z^2}{1+z^2}; \qquad \frac{dG}{dz} = \exp\frac{z^2-1}{z^2+1}.$$

Hence differentiating logarithmically, we see that

$$(z^2+1)^2\frac{d^2F}{dz^2} + 4z\frac{dF}{dz} = 0; \quad (z^2+1)^2\frac{d^2G}{dz^2} - 4z\frac{dG}{dz} = 0. \tag{5.1}$$

Consider the differential equation for $F(z)$. If we make the substitution $w = z^2/(z^2+1)$, this differential equation becomes

$$w(1-w)\frac{d^2F}{dw^2} - (2w^2 - w - \tfrac{1}{2})\frac{dF}{dw} = 0. \tag{5.3}$$

The equation has 0 and 1 for regular points and ∞ for an irregular point. Now drawing upon the results of the theory of second order linear differential equations,[2] we see that if (5.3) is regarded as obtained by confluence from a differential equation with only regular points, the initial equation must have had five or more regular points. By actual trial, we find it is impossible to derive (5.3) from a differential with exactly five regular points. But all of the elementary functions of mathematical physics may be derived as solutions of confluent

[1] For convenience in printing, we hereafter write z for Z.

[2] See for example Whittaker and Watson, *Modern Analysis* Chap. X; or Ince, *Ordinary Differential Equations*, Chap. XX.

forms of such an equation. Hence $F(z)$ cannot be expressed in finite form by means of such functions.

Neither can $F(z)$ be expressed by means of elliptic integrals of the first or second kinds, for such integrals have no essential singularity in any part of the plane. A precisely similar argument holds for $G(z)$.

6. We are thus driven to seek series representations of $F(z)$ and $G(z)$ in the range in which we are interested. One such series is immediately obvious; namely an expansion about the point $z=1$ in ascending powers of $z-1$.

We have by Taylor's theorem

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!}(z-1)^n, \quad G(z) = \sum_{n=0}^{\infty} \frac{G^{(n)}(1)}{n!}(z-1)^n,$$

the radius of convergence of both series being $\sqrt{2}$ in accordance with section 3, (b), (c). On writing $z=e^{i\theta}$, we find that $z-1=2e^{(\pi+\theta)i/2}\sin(\theta/2)$. Hence

$$(6.1)\qquad \begin{aligned} F(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{F^{(n)}(1)}{n!} e^{ni/2(\pi+\theta)}\left(2\sin\frac{\theta}{2}\right)^n, \\ G(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{G^{(n)}(1)}{n!} e^{ni/2(\pi+\theta)}\left(2\sin\frac{\theta}{2}\right)^n, \end{aligned} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Since $F(z)$ and $G(z)$ are real when z is real and greater than -1, the constants $F^{(n)}(1)$ and $G^{(n)}(1)$ in (6.1) are all real. Hence on combining these formulas with (4.1), we find that

$$(6.2)\quad K_1(\theta) = \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{F^{(n)}(1)+G^{(n)}(1)}{n!} \sin\frac{n}{2}(\pi+\theta)\left(2\sin\frac{\theta}{2}\right)^n, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

with similar formulas for $K_2(\theta)$, $K_3(\theta)$ and $K_4(\theta)$.

To calculate the constants $F^{(n)}(1)$ and $G^{(n)}(1)$, we differentiate the equations (5.1) $n+2$ times and set $z=1$. We thus obtain the recursion formulas

$$(6.3)\qquad \begin{aligned} F^{(n+4)}(1) &= -(2n+5)F^{(n+3)}(1) - (n+2)(2n+3)F^{(n+2)}(1) \\ &\quad - (n+2)(n+1)nF^{(n+1)}(1) - \frac{(n+2)(n+1)n(n-1)}{4}F^{(n)}(1); \\ G^{(n+4)}(1) &= -(2n+3)G^{(n+3)}(1) - (n+2)(2n+1)G^{(n+2)}(1) \\ &\quad - (n+2)(n+1)nG^{(n+1)}(1) - \frac{(n+2)(n+1)n(n-1)}{4}G^{(n)}(1). \end{aligned}$$

On setting $n=-2, -1, 0, 1, \cdots$ in these formulas, we find from (5.2) that[1]

[1] These coefficients have been checked from (3.1) by expanding $\exp\pm(z^3-1)/(z^2+1)$ in ascending powers of $z-1$ up to the terms of order $(z-1)^7$, and integrating term by term.

$F^{(0)}(1) = 0$; $F^{(1)}(1) = 1$; $F^{(2)}(1) = -1$; $F^{(3)}(1) = 2$; $F^{(4)}(1) = -4$; $F^{(5)}(1) = 4$; $F^{(6)}(1) = 34$; $F^{(7)}(1) = -374$; $F^{(8)}(1) = 2498$.

$G^{(0)}(1) = 0$; $G^{(1)}(1) = 1$; $G^{(2)}(1) = 1$; $G^{(3)}(1) = 0$; $G^{(4)}(1) = -2$; $G^{(5)}(1) = 4$; $G^{(6)}(1) = 6$; $G^{(7)}(1) = -74$; $G^{(8)}(1) = 190$.

On substituting these values in the first nine terms of (6.2), we obtain the series for $K_1(\theta)$ given in section 1.

7. Let $z = e^{i\theta}$ be a fixed point on the unit circle to the right of the y axis, and C a circle of radius $\rho < \sqrt{2}$ about the point $z = 1$ including the point $e^{i\theta}$. Then

$$F(e^{i\theta}) = \frac{1}{2\pi i}\int_C \frac{F(w)dw}{w - z}, \tag{7.1}$$

where w denotes a complex current co-ordinate upon the circle C.

From the identity

$$\frac{1}{w - z} = \frac{1}{w - 1} + \frac{z - 1}{(w - 1)^2} + \cdots + \frac{(z - 1)^n}{(w - 1)^{n+1}} + \frac{(z - 1)^{n+1}}{(w - 1)^{n+1}(w - z)},$$

we obtain

$$F(z) = c_0 + c_1(z - 1) + \cdots + c_n(z - 1)^n + \Re_n$$

where $c_k = F^{(k)}(1)/k!$ $(k = 0, \cdots, n)$ and

$$\Re_n = \frac{(z - 1)^{n+1}}{2\pi i}\int_C \frac{F(w)dw}{(w - 1)^{n+1}(w - z)}. \tag{7.2}$$

Now

$$|z - 1| = 2\sin\frac{|\theta|}{2}, \quad |w - 1| = \rho, \quad |w - z| \geqq \rho - |z - 1| = \rho - 2\sin\frac{|\theta|}{2}$$

and by 3 (d),

$$|F(w)| \leqq \rho \exp\frac{\sqrt{\rho^4 + 4} + \rho^2}{2(2 - \rho^2)}.$$

Hence from (7.2),

$$|\Re_n| \leqq \left(\frac{2\sin\frac{|\theta|}{2}}{\rho}\right)^{n+1} \frac{\rho^2}{\rho - 2\sin\frac{|\theta|}{2}} \exp\frac{\sqrt{\rho^4 + 4 + \rho^2}}{2(2 - \rho^2)}.$$

The inequality for the remainder in the series for $G(z)$ is precisely the same, save that the numerator of the exponential is replaced by $\sqrt{\rho^2+4}-\rho^4$.

A somewhat better inequality when n is large may be obtained by integrating the right side of (7.2) by parts before obtaining the dominant. It gives

$$|\Re_n| \leqq \frac{1}{n}\left(\frac{2\sin\frac{|\theta|}{2}}{\rho}\right)^{n+1} \frac{2\rho^2\left(\rho+\sin\frac{|\theta|}{2}\right)}{\left(\rho-2\sin\frac{|\theta|}{2}\right)^2} \exp\frac{\sqrt{\rho^4+4}+\rho^2}{2(2-\rho^2)}.$$

If we take $\rho^2=3/2$, $\theta=\pi/4$ in the first inequality, we obtain

$$|\Re_n| < (\tfrac{5}{8})^{n+1} \times 51.08 = .0042 \text{ for } n = 19.$$

The second inequality gives

$$|\Re_n| < \frac{1}{n}\left(\frac{5}{8}\right)^{n+1} 438 = .0019 \text{ for } n = 19.$$

If θ is quite small, we may take $\rho=1$, obtaining

$$|\Re_n| < \left(2\sin\frac{|\theta|}{2}\right)^{n+1} \frac{e^{(\sqrt{5}+1)/2}}{1-2\sin\frac{|\theta|}{2}} \text{ for } F(z)$$

and

$$|\Re_n| < \left(2\sin\frac{|\theta|}{2}\right)^{n+1} \frac{e^{(\sqrt{5}-1)/2}}{1-2\sin\frac{|\theta|}{2}} \text{ for } G(z).$$

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

By T. C. BENTON, Pennsylvania State College

1. *Introduction.* After teaching the subject of linear differential equations to sophomore students a number of times, it has seemed to the author that the customary methods of assuming the correct answers and then verifying them are highly unsatisfactory from a pedagogical viewpoint. The student always asks how the form of solution used was obtained in the first place. Also the better student is left with the feeling, that except for a lucky guess, there is no way to obtain the solution of similar problems. It is the purpose of this development of the subject to present a method in which every step is forced—a method in which there is no guesswork at all. The actual work is all of well known character but the fact that the general methods of the higher theory of differential equations work out in such a simple way for the elementary cases seems worthy of attention

2. *General Theory* of the solution of:

$$(A) \qquad \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

P, Q being functions of x or constants.

Definition—If $y=u(x)$, $y=v(x)$, $y=w(x)$, etc. are solutions of (A) such that $c_1u+c_2v+c_3w+\cdots\neq 0$ for all values of x unless $c_1=c_2=c_3=\cdots=0$, we will say that $u(x)$, $v(x)$, $w(x)$, etc. are independent solutions. By this definition, if u and v are independent solutions neither u nor v is identically zero.

Theorem 1.—If u, v, w, $\cdots$ are solutions of (A) the function $y=c_1u+c_2v+c_3w+\cdots$ is also a solution.

This follows by direct substitution in the equation.

Theorem 2.—If u and v are two independent solutions of (A), and w is any solution not identically zero, there must be two constants k_1 and k_2 not both zero, such that $w\equiv k_1u+k_2v$.

Proof—Let u and v be two independent solutions of (A). Then,

$$u'' + Pu' + Qu = 0 \tag{1}$$

$$v'' + Pv' + Qv = 0, \tag{2}$$

where the primes denote differentiation with respect to x. Multiplying (1) by v and (2) by u and subtracting we obtain:

$$u''v - uv'' + P(u'v - uv') = 0. \tag{3}$$

Now by direct differentiation $d(u'v - uv')/dx = u''v-uv''$, so (3) becomes $d(u'v - uv')/dx = -P(u'v-uv')$. Then since, by the independence of u and v, $uv'-u'v\not\equiv 0$, we have

$$\frac{d(u'v - uv')}{(u'v - uv')} = -Pdx.$$

Solving this we obtain:

$$u'v - uv' = Ce^{-\int Pdx}, \quad C \neq 0, \tag{4}$$

which gives a relation between any two independent solutions and their derivatives.

By a similar process, if we assume that $u'w-uw'\not\equiv 0$, we obtain:

$$u'w - uw' = De^{-\int Pdx}, \quad D \neq 0; \tag{$\bar{4}$}$$

and if we admit the case $D=0$, $(\bar{4})$ holds for any pair of solutions. Hence multiplying (4) by D and $(\bar{4})$ by C and subtracting it follows that

$$(Dv - Cw)u' - (Dv' - Cw')u = 0.$$

If $Dv-Cw\equiv 0$, the conclusion of the theorem is satisfied with $k_1=0$, and $k_2=D/C$. If $Dv-Cw\not\equiv 0$, we can divide by $(Dv-Cw)u$, obtaining

$$\frac{u'}{u} = \frac{Dv' - Cw'}{Dv - Cw}.$$

Hence $\log u+\log k=\log (Dv-Cw)$; but then, since $C\neq 0$, $w=(-k/C)u+(D/C)v$, which proves the theorem for this case.

3. *Constant Coefficients.* Let the constants p, q replace the functions P, Q, giving the equation:

$$u'' + pu' + qu = 0. \tag{B}$$

In this equation let $z=u'/u$. Then $u'=zu$, $u''=z'u+zu'=z'u+z^2u$, so (B) becomes

$$(z' + z^2 + pz + q)u = 0. \tag{5}$$

If u is not identically zero, we must have $z'=-(z^2+pz+q)$. This first order equation might be solved by separating the variables; but as the different cases of integrating dz/z^2+pz+q are somewhat complicated, a much simpler attack is to notice that a constant value for z means $z'=0$ and therefore $z^2+pz+q=0$. Now this equation has two roots if we admit complex numbers, so there are always two constants $z=k_1$, $z=k_2$, which satisfy (5). But for these values of z, we obtain

$$u = C_1e^{k_1x} \text{ or } u = C_2e^{k_2x}.$$

Case 1: Suppose $k_1\neq k_2$. We will now show that the two solutions obtained for u are independent. For if not, let C and D be constants not both zero such that $Ce^{k_1x}+De^{k_2x}=0$ for all values of x. We have for $x=0$, $C+D=0$; and for $x=1$, $Ce^{k_1}+De^{k_2}=0$. These two equations imply that $C=0$ and $D=0$. As this contradicts our premise, the two solutions must be independent.

It then follows that the general solution of (B) is by theorem 1,

$$u = C_1e^{k_1x} + C_2e^{k_2x}.$$

Case 2: Suppose $k_1=k_2$. Then we have $p^2=+q$ and $k_1=k_2=-p/2$. By equation (4) we have, for independent solutions u and v,

$$u'v - uv' = e^{-\int p\,dx}.$$

Using $u=Ce^{-(p/2)x}$, we have

$$-(p/2)Ce^{-(p/2)x}\cdot v - Ce^{-(p/2)x}\cdot v' = e^{-px+h}.$$

Since h is arbitrary we can choose it so that $e^h=C$. Then dividing by $-e^{-(p/2)x}$, the result is $v'+(p/2)v=-e^{-(p/2)x}$. Applying the usual formula to solve this linear equation of the first order we obtain $v=Ke^{-(p/2)x}-xe^{-(p/2)x}$. Since we wish only the simplest solution which is independent of $e^{-(p/2)x}$ we can take $K=0$ without any loss of generality and use $v=-xe^{-(p/2)x}$ as the other solution, proving the independence by much the same method as in case 1.

Hence in this case the general solution is:

$$u = e^{-(p/2)x}(C_1 + C_2x).$$

4. *Comments.* The equation (4) of theorem 2 part 1 is the relation which is

usually called Abel's Identity. Its use in the case of equal roots seemed to make students feel that there was much more reason for the form of the second particular solution.

There is, of course, no theoretical objection on a logical ground to the usual presentation but it seems to the author that each step in the proofs here given is almost forced upon one, and that this is a big factor in giving the student assurance that there is nothing amiss with his work.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

Vector Analysis. By H. B. Phillips. New York, John Wiley & Sons, 1933. viii +236 pages. $2.50.

This book gives an account of three dimensional vector analysis with many applications to electromagnetic theory and hydrodynamics. The treatment is sound and careful, and each of the first eight chapters carries a score or more of exercises and problems. Chapter 9 is devoted to the theory of the retarded potential and chapter 10 to a rather outmoded account of linear vector functions (dyadics, etc.) There is no particular claim to novelty either in subject matter or in treatment, and the book will undoubtedly be useful to students.

However, in writing this evaluation, the reviewer cannot forget a review by Clifford of a book of Booth's in which he referred to the author as serenely indifferent to the new-fangled methods which were tending to disturb the complacency of the older British School. So many interesting things have happened in the field of vector analysis since, say 1913, when the present book might well have been written, that it seems a shame not to let the student know about them. But whilst expressing this opinion, we recognize that we have been asked to tell about a book which *has* been written, and not about one which *might* be written.

The present actual book may be recommended as one of the best accounts of the vector analysis of Gibbs, and of its applications, to problems of mathematical physics.

F. D. MURNAGHAN

The Theory of Ruled Surfaces. By W. L. Edge. Cambridge University Press, 1931. x+324 pp. $7.00.

This work on the theory of algebraic ruled surfaces in Euclidean space of three dimensions, while not introducing any methods not previously used, has made use of two of the most important of these methods in conjunction to classify the quartic, quintic, and sextic ruled surfaces. No such classification so far published has been shown to be complete, and this enumeration is no excep-

tion, in spite of the author's claim. In fact, surfaces other than those listed have been mentioned by other writers; in particular, for sextic surfaces, compare the series of papers of Virgil Snyder in the American Journal of Mathematics, vol. 25 (1903) and vol. 27 (1905). This fact does not mean, however, that the work fails to be a real contribution to mathematical literature; for, to quote from the author's preface, "there exists at present no work, easily accessible to English readers, which tests the application of the general ideas here employed in anything like the same detail. One might mention especially the use of higher space and the principle of correspondence, and these two ideas are vital and fundamental in all modern algebraic geometry."

In the first chapter, the author recalls to the reader the salient points of space curve and surface theory, and discusses the theory of correspondence of Chasles-Cayley-Brill, which later is of such great value. He also outlines the two methods which he intends to use in classifying ruled surfaces. The first of these employs the Klein representation of lines in space as points of a quadric four-dimensional hypersurface Ω in five-space. Thus a ruled surface f is represented by a curve C on Ω, and the properties of f are related to properties of C. The second method, one which works equally well for ruled surfaces in n-space, is to consider the given surface as the projection of a normal ruled surface in a space of higher dimensionality. This latter method is particularly valuable for the study of rational surfaces, and often aids in setting up the given surfaces as the totality of lines joining corresponding points of two directrix curves. Here the principle of correspondence comes into play. In the last analysis, the classification is made according to the nature of the double curve and the bitangent developable.

The quartic, quintic, and sextic varieties are treated in detail in separate chapters, and one chapter is devoted to the developables. The book proper closes with a classification table summarizing the author's results, which is followed by a note concerning the intersections of two curves on a ruled surface.

It seems to the reviewer that an index would have added to the value of the book as a work of reference, preferably a general index, but at least a page reference list in connection with the tables at the close of the work.

The book is well written and well printed, and has the decided advantage of being one of the first compilations in one volume of the outstanding known facts in this interesting field.

ROBIN ROBINSON

Numerology. By E. T. Bell. New York, The Century Co., 1933. vii+187 pages. $2.00.

In order that the meaning of the term *numerology*, or number-mysticism, may be clear at the outset, the author in Chapter I gives three illustrations: *Hollywood in the Middle Ages*, concerning (as he says) Hollywood and theology; *Vibrations, Bulls and Bears*, featuring a stock broker who changed his name to harmonize with that "inner and insensible number" of which his "vibrations"

are "the outward and sensible manifestations"; and *A Modern Miracle* of physics (dating from February, rather than March, 1932 as stated by the author). Later on he writes, "Whoever believes mathematical theories of the universe to be anything more than convenient maps that may be radically revised or torn up at any moment is a numerologist. A scientific worker who holds no such belief is a scientist. There is of course no stigma on either term." Numerology would seem to bear a relation to mathematics much like that of astrology to astronomy or of alchemy to chemistry.

The following list of chapter titles, together with the interpolated remarks, should adequately suggest the trend of the discussion: I. *Through Hollywood to Science*; II. *Numbers and Marriage*, introducing Plato's "nuptial number", 60^4; III. *Recurrent Nightmares*, concerning recurrence in decimals and in physical phenomena; IV. *500 B. C.* and V. *In the Beginning*, largely devoted to the numerology of Pythagoras, "colossus" of the art; VI. *Ancient and Modern*, with comment on "recent attempts to numerologize art" and a consideration of perfect numbers, "masses at infinity," and numerological chemistry; VII. *Sacred and Profane Numerology*, where one finds "The populace openly declared that sacred numerology was tripe" and the author makes his bow to intelligence testing under the sub-title *Modern Beasting*; VIII. *A Great Dream*, in which one reads "The most comprehensive dream ever dreamed by our race is short: *The Cosmos is isomorphic with pure mathematics*," and the writer pays his respects to extrapolation; IX. *Ulysses*, with sub-titles *Numerologized Reason* and *Numerologized Mathematics*. An Appendix includes comments on certain types of questions frequently found in intelligence tests, a simple paradox, a few numerological examples, and several easily stated mathematical problems (some of them unsolved) chosen mainly from number theory.

We have been told that "Life is real! Life is earnest!" To most of us the first proposition is acceptable, although to a few philosophers it may offer the stimulus to construct a *word-hash*. For the majority of us life is both earnest and serious. Especially to mathematicians is mathematics a very serious matter. Here, however, we have a mathematician of high repute who does not take all that bears the label of mathematics (or science) too seriously. We can imagine him saying "A 'word-hash' cannot be mapped upon mathematics, or any part thereof; it can be mapped only on a 'number-hash'—enter numerology."

Numerology is written in a breezy and entertaining style, and no doubt will soon be offered as evidence that California produces, along with choice prunes and other things, bigger and better satires. Superficially the book is non-technical; it will be found amusing and informative by almost any layman with an interest in science. Most professional scientists, also, will find the book interesting; some few of them may be incensed, especially if they interpret the section entitled *A Request* in Chapter VII as a not very subtle inference that among American scientists Professor Bell himself is alone free from numerological tendencies.

C. R. ADAMS

Solid Geometry. By W. H. Macaulay. Cambridge University Press, 1930. xii+303 pages. $4.75.

The book is a text on solid analytic geometry, using rectangular coordinates, considering only real loci, and making no use of the notion of points at infinity. Besides the usual topics through quadric surfaces, there are chapters on twisted curves and developables, the general theory of surfaces, spherical trigonometry; and then four chapters for the purpose of connecting the theory of solid analytics with some of its applications, namely moments of inertia, strain, stress, and vector distributions. There is also an appendix containing a few miscellaneous notes and examples.

The author uses his own ideas in places. When classifying the quadric surfaces, for instance, he first forms all quadrics of revolution by rotating the various conics about their axes, then subjects them to a homogeneous strain, then to a limiting process which carries a system of parallel elliptic or hyperbolic sections into parabolas. This process gives us a conception of how the surfaces look. He also takes the general equation of second degree and simplifies it by rotation and translation. Again, in representing a vector distribution, instead of picturing an arrow of determined length and direction emanating from each point of space, he pictures space as filled with curved lines whose direction gives the direction of the vector, and so packed that the number per unit area cutting a small piece of surface equals the component of the vector perpendicular to the surface. He has coined a convenient word, "conformable," to denote the relation between two triads of rectangular axes which can be made to coincide by a continuous motion, without a reversal of direction. We had been saying, "both right-handed, or both left-handed." A knowledge of calculus through partial differentiation and Taylor's series is presupposed.

The style of the book is decidedly English. The exposition is very lucid and scholarly, and the text free from errors. There are few figures, and no special devices for catching the eye and calling attention to the main facts and formulas. The equations are not even numbered for later reference. I should say that the book would not be so good as a reference work, but would be excellent for cultural purposes.

D. F. BARROW

Linguistic Analysis of Mathematics. By A. F. Bentley. Bloomington, Indiana, Principia Press, Inc., 1932. xii+315 pages. $3.00.

The language of mathematics here treated includes not merely mathematical symbols but also "those immediately surrounding forms of expression and assertion through which the symbols are developed, communicated and interpreted." Language, with all of its imperfections, affords the only means of communicating a critical analysis of language itself (p. viii). "Development must therefore be precarious: it must be in part impressionistic . . . . Only as an island in this sea of linguistic confusion may a small region of precision be established."

The author insists that the foundations of mathematics must be sought in mathematics itself and that no reliance should be placed on any external source, such as logic for instance (pp. 166, 308), since (p. 8) "mathematics is the safest and most certain knowledge the world possesses or thus far has had prospect of possessing." The author realizes (p. 49) that "the reader whose habits of work require him to commence with firm definition and classification will find herein nothing but disappointment." His point of view is strictly empirical. He does not try to give satisfaction to the mathematician (see p. 60); he seeks rather to disturb him in the hope of uncovering "possible seeds of fertility."

There is too much lack of precision in the book for the reviewer to be able to extract from it any very useful definitive results. The author himself speaks of it (p. 311) as "a record of exploration, not an achieved formulation." The most clearly written part is Chapter XI (pp. 181–211) on the denumerability of decimals; but the principal part of this chapter is demonstrably invalid, in particular that part in which the author undertakes to show that the totality of decimals is denumerable. A certain satisfaction with vagueness (see p. 248 for instance) characterizes a considerable part of the exposition. The reviewer believes that the exploration should have been carried to a stage of more definite achievement before publication. The work is too tentative in character to call for much attention from other thinkers. Unless the author himself can more fully justify his method and procedure by results achieved he is not entitled to hope that it will have any very useful effect upon mathematical thinking.

R. D. CARMICHAEL

Economic Control of Quality of Manufactured Product. By W. A. Shewhart. New York, D. Van Nostrand Company, Inc., 1932. viii+501 pages. $6.50.

Upon reading the title of this book the natural question is, "Why review it in a mathematical journal?" It is true that the book is not written primarily for mathematicians nor for students of mathematics but for the manufacturer and the engineer. Nevertheless, it is interesting to the mathematician be he pure or applied. No matter how often we may tell the story of the mathematician who took great pleasure in asserting that his own special interest in mathematics had no possible applications, we take more or less pride in saying that sooner or later any branch of mathematics will find an application.

What is this problem of economic control? To answer the question from the book, let us consider a fountain pen point. A maker of such pen points turns out thousands per year. He has in his mind a point which he calls standard. Then he tries to make pens which conform to this standard. This seems a simple problem, but we all know the variability of pens of the same make, price and style as well as the variability of the hands of the persons using them. Unknown or chance causes have their effect. If we cannot make all pens to the exact pattern of the standard pen, the next best thing is to know how a given shipment may vary from this standard. The manufacture of an article is said to be controlled

when we can predict the variations from the standard which is set up—that is, when we can state the probability that the measure of quality will fall within given limits. Put more specifically, each maker of fountain pen points will have associated with his pen certain data, which put into geometrical form is the ordinary frequency histogram.

The structure of the book has mathematical form, in that three postulates underlie its theory of control. In shortened form these postulates are

1. All chance systems of causes are not alike.
2. Constant systems of chance causes exist.
3. Assignable causes of variation may be found and eliminated.

By weeding out assignable causes of variation the product is considered as controlled and the advantages to the manufacturer are

1. Reduction in the cost of inspection.
2. Reduction in the cost of rejection.
3. Attainment of maximum benefits from quantity production.
4. Attainment of uniform quality even though the inspection test is destructive.
5. Reduction in tolerance limits where quality measurement is indirect.

The book is made up of seven "Parts" and appendices. The above is a short summary of Part I which is introductory. In Part II the word "quality" used in the introduction is discussed and defined. The concept of variability used in the definition of quality leads naturally to the main features of this Part, which are those of a good text book on the fundamentals of statistics, taking up the basic notions of grouping, measures of central tendency, dispersion, correlation, skewness, and in general, the methods of reducing large numbers of observations of quality to a few simple statistics which contain the essential information.

In Part III the relations with the theory of probability are taken up. Statistical laws are stated to be the frequency distributions arising from the general law of large numbers. Various systems of frequency distributions are discussed. An interesting idea is that of "Maximum Control," which is defined as the condition reached when the chance fluctuations in a phenomenon are produced by a constant complex of a large number of chance causes in which no cause produces a predominating effect. This notion brings to mind the Galton board and the next thought is on normality, but the author takes pains to show that normality of distribution is not a necessary condition for maximum control. "Most distributions exhibiting control have been found to be sufficiently near normal to be fitted by the first two terms of the Gram-Charlier series." This point is interesting to the reviewer, for at first glance it would appear that as a prescribed standard approached perfection the distribution would develop considerable skewness.

Part IV is a readable chapter on sampling written from the standpoint of the statistician who must look upon the distribution function of a statistic as a working tool. The chapters discuss the usual statistics including the correlation coefficient. Much of the very recent advancement which has been made in

sampling theory is summarized. The establishment of tolerance ranges and the setting up of standard of quality are shown in Part V to be problems based in general on the three statistics, arithmetic mean, standard deviation and relative frequency of defectives.

The determination of lack of control, that is, the establishment of an efficient method for detecting the presence of a cause of variability other than chance is the subject matter of Part VI. The first of two main problems to be considered is that of establishing an efficient method for determining when an observed sample is such that it is unlikely that it came from a complex of causes characterized by the prescribed standard distribution. The second and more complex problem is that of finding out whether the observed data arise from some not specified constant system of causes. Five criteria which have been found to work successfully are given and discussed. Part VII on quality control in practice gives a summary of the fundamental principles underlying the theory of control and an outline of the method of attaining control of quality from the raw material to the finished product. There are two appendices of "Resultant effects of constant cause systems" and on "Experimental results." Appendix III is an extensive classified bibliography.

This brief outline of the main points of Dr. Shewhart's work will give but a sketchy idea of its contents. There is great wealth of illustration by means of problems that have actually arisen in practical work and many statistical processes are illustrated by problems that will be valuable supplements to the ordinary text book discussions. Dr. Shewhart's book will tend to increase the growing use of the term, "Bell Telephone School," in connection with statistics.

A. R. CRATHORNE

Analytic Geometry. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1933. xii+296 pages. $2.25.

This book contains a very thorough treatment of the subject matter of Plane Analytic Geometry plus a chapter on Determinants. It is rigorous, but I believe certain portions could be treated in a more simple fashion without losing rigor.

Orthogonal projections are introduced in the early part of the first chapter and, as the author points out in the preface, are used extensively throughout. There is no confusion regarding the definition of the inclination of a line. The slope of a line is treated in the usual way and in addition there is a non-trigonometric treatment. Polar coordinates are introduced in the first chapter.

Functional notation is introduced in the first of the two chapters on loci and several theorems are developed regarding changes in the form of the equation of a locus and possible changes resulting in the loci themselves.

In the chapter on the straight line no mention is made of Hesse's normal form and no use is made of it in deriving the formula for the perpendicular distance from a line to a point. I regret this. Parametric equations of the straight line are introduced in this chapter.

The circle is handled very nicely.

The general conic is defined and the equations of the three types developed from the equation of the general conic. For this purpose the author introduces translation of axes and derives his equations by means of projections. The coordinates of a focus and the equation of a directrix are derived directly for the three types of conics from the equation of the general conic and its consequences, but the details seem a bit formidable for one approaching the subject for the first time.

In the next two chapters, one on tangents, normals and chords of contact, and the other on diameters, poles and polars, extensive use is made of the equations of the straight line in parametric form.

There follow two short chapters, one devoted to properties of conics and one to rotation of axes. The equations for rotation are derived by the use of polar coordinates.

We have then a much longer chapter in which there is a detailed study of the general equation of the second degree with a discussion of invariants in which use is made of determinants.

There is a short chapter on Higher Plane Curves, followed by a good treatment of the elements of determinants.

The book is well written and is suitable for use in a very complete course in Plane Analytic Geometry but would not, I believe, be very practical for a freshman course of which the latter part is devoted to an introduction to the Calculus.

A. H. SPRAGUE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Oklahoma

Our chapter reports the following officers for the year 1931–1932: C. E. Springer, Director, Dora McFarland, Vice Director, Dorothy Huff, Secretary, Carl Ransbarger, Treasurer, Professor J. O. Hassler, Librarian. They were elected May 31, 1931, by ballot.

Of the thirty-seven active members, fifteen are members of the faculty, six are graduate students, and sixteen are undergraduates. An election of members was held on March 10, 1932 and fifteen new members were admitted. These fifteen were formally initiated at a banquet given April 8, 1932.

Regular meetings were held on the second and fourth Thursdays of each month. During the year the following papers and topics were discussed:

October 22, 1931: "Fourier series" by Mr. Carl Ransbarger.
November 5, 1931: "Matrices and determinants" by Mr. Woodward and Miss Huff.
November 19, 1931: "Fundamental properties of determinants" by Miss Wynne and Mr. Humphrey.
December 10, 1931: "Mathematical aspects of philosophy" by Dr. N. A. Court.
January 14, 1932: "Integral equations" by Mr. Stephen Brixey.
February 25, 1932: "The expanding universe" by Mr. Whitney; "Numbers which are equal to the sum of their aliquot parts diminished by the number of aliquot parts" by Mr. John Brixey.
April 21, 1932: "Gamma functions" by Miss McGregor and Miss Gassett; "The trisection of an angle" by Miss Dorsett.
May 12, 1932: "Dynamic aspects of the theory of moving axes" by Mr. Dorsett.

The business meetings of the year were:
March 10, 1932: Election of new members.
May 19, 1932: The election of officers for the year 1932–1933.

The social meetings of the year were as follows:
October 11, 1931: A tea at the Faculty Club entertaining the mathematics majors.
October 29, 1931: A picnic at the Norman Country Club.
April 19, 1932: A luncheon given at the Faculty Club in honor of Dr. W. D. MacMillan of the University of Chicago.

DOROTHY HUFF, *Secretary*

Pi Mu Epsilon of Syracuse University

This chapter had a successful year under the guidance of Paul D. Jose, the Director. The chapter has expanded and now has 61 active members and 28 faculty members. The affairs of the fraternity were handled by the following officers: Paul D. Jose, Director; Jane Armstrong, Vice Director; John E. Backman, Treasurer; Mazie Chapman Lloyd, Secretary.

These officers were elected by ballot May 13, 1931. Upon the resignation of Mazie Lloyd, the position of Secretary was filled by Helen Heineman who was elected by acclamation February 13, 1932.

On December 15, 1931, seventeen new members were welcomed into the chapter. After the initiation we had a social hour. At our Spring banquet on May 11, 1932, six more persons were initiated.

At the first meeting of the year, October 28, 1931, Dr. William P. Graham spoke on "Einstein's theory of relativity." This interesting talk was followed on November 17, 1931 by a meeting under the management of the Engineers. At this meeting John Norton spoke on "The relation of mathematics to chemical engineering"; Martin Hogan showed the relation between mathematics and electrical engineering; Edward Backman told how mathematics is involved in the building of bridges; and Joseph Carroll connected mathematics and mechanical engineering. Three Liberal Arts students had charge of the next program meeting of February 16, 1932. Helen Heineman spoke on "Egyptian mathematics"; Francis Persons on "The life of Newton"; and Helen Noble on "A few of the relations between mathematics and physics." Several other meetings were planned which unfortunately had to be postponed when our campus was buried in snow. A meeting was held on April 21, 1932 but the program had to be omitted because of the length of the business meeting.

In the Fall, Pi Mu Epsilon had a picnic at one of the near-by State Parks, to which were in-

vited juniors and seniors who were majoring in mathematics. A similar picnic was held on May 18, 1932.

We have had a busy and interesting year and hope to continue to expand as we have this year.

HELEN HEINEMAN, *Secretary*

Pi Mu Epsilon of The State College of Washington

The year 1931–1932, the first year for our chapter, was a successful one, particularly from the standpoint of the number of interesting papers that were presented. All of the meetings except the business meetings were held jointly with the Newtonian Society which includes freshmen and others who are not members of Pi Mu Epsilon. The starred names are members of Pi Mu Epsilon. The papers were given in the following order:

October 28, 1931: "Nomography" by *Professor C. A. Isaacs.

November 12, 1931: "Descriptive properties of point sets" by *Miss Jane Secrest; "Characteristic sets of points" by Mr. Don Jenne.

December 2, 1931: "The base of our number system" by *Miss Thelma Peterson; "The measure of sets of points" by *Mr. Ward Crowley.

January 6, 1932: "Two geometric fallacies" by Miss Claire Lasater; "Massie's method of the trisection of an angle" by Miss Laura Colpitts; "Algebraic fallacies" by Miss Marianne Hawley; Mr. R. Summers exhibited some algebraic and other fallacies; Miss Doris Lee showed a linkage for the trisection of an angle.

January 21, 1932: "Telescope making" by *Professor H. H. Irwin.

January 27, 1932: "Mars" by Miss Emma Pell; "Astronomical distances" by *Professor E. C. Colpitts.

February 18, 1932: "Flatland" by *Miss Ruth Peterson; "Projective geometry" by *Mr. J. R. Vatnsdal.

March 3, 1932: "Impossibility of angle trisection" by *Mr. L. G. Butler; "Magic squares" by Mr. Sam Rausch.

March 17, 1932: "Invariants" by *Mr. J. Biggerstaff; "Meteorology" by Mr. Don Jenne.

March 31, 1932: "Continued fractions" by *Miss Mildred Hunt; "What mathematics can do for you" by Mr. Stephen Cristopher.

A banquet on April 15, 1932 was held jointly with the Newtonian Society.

During the first semester of 1931–1932, five new members were initiated and during the second semester five more were given membership.

The officers for the year were: Miss Jane Secrest, Director; Miss Grace Leyde, Secretary; Mr. C. A. Isaacs, Treasurer.

GRACE LEYDE, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of The George Washington University

The officers for 1931–1932 were: Dr. F. E. Johnston, President; Mr. Albert Wertheimer, Secretary.

The meetings and programs were as follows:

October 12, 1931: "On the Galois theory of equations" by Dr. F. E. Johnston.

October 28, 1931: "On certain properties of the Beta function" by Daniel B. Fisher.

November 9, 1931: "A generalized error function" by Albert Wertheimer.

November 23, 1931: "Ovals of constant breadth" by Dr. P. J. Federico.

December 7, 1931: "On a problem in permutations and combinations" by A. Sinkov.

January 11, 1932: "On the geometry of paths" by Dr. James H. Taylor.

February 23, 1932: "Integration" by Professor E. R. Hedrick of the University of California at Los Angeles.

March 7, 1932: "A problem in the theory of transformations" by Dr. Tobias Dantzig of the University of Maryland.

March 21, 1932: "Integral equations" by Michael Goldberg.
April 11, 1932: "Product series" by Dr. Florence M. Mears.
April 22, 1932: "The calculus of variations and quantum theory" by Professor G. A. Bliss of the University of Chicago.
May 9, 1932: "The problem of mathematical consistency *im kleinem* and *im grossen*" by Captain E. E. Hagler.

Albert Wertheimer, *Secretary.*

Club Topics

1933 as a Centennial Year in the History of Mathematics

By W. C. Eells, Stanford University, California

In continuation of previously published lists (See this Monthly, vol. 39 (1932), pp. 298–299, for a list of 1932 centennial events, and for references to previous volumes for corresponding lists from 1925 to 1931) of centennial dates in the history of mathematics, the following group of important 1933 centennial dates is presented.

A.D. 833. Death of the Caliph, al-Mamun, Arabian astronomer and patron of learning who supervised two geodetic surveys of Mesopotamia and under whose direction was completed the translation of Euclid's *Elements* into Arabic.

A.D. 1533. Posthumous publication of "Regiomontanus" (Johann Müller) *De triangulis omnimodis libri V*, the first work that may be said to have been devoted solely to trigonometry, written about 1464.

A.D. 1533. Appearance in Europe of the Greek text of Euclid's *Elements*, a publication of the *Editio Princeps* by Grynaeus.

A.D. 1533. Death of Köbel, German author of the popular arithmetic *Rechenbiechlin* (1514) which passed through at least twenty-two editions.

A.D. 1633. Birth of Mei Wen-ting, important Chinese historian of mathematics.

A.D. 1633. Birth of Hudde, Dutch mathematician, author of an ingenious rule for finding equal roots.

A.D. 1633. Posthumous publication of Briggs's *Trigonometria Britannica.*

A.D. 1733. Daniel Bernoulli leaves his professorship of mathematics at St. Petersburg for a professorship at Basel, Switzerland.

A.D. 1733. Euler succeeds Bernoulli as professor of mathematics at St. Petersburg.

A.D. 1733. Death of Saccheri, Jesuit father of Milan, who in the year of his death wrote *Euclides ab omni naevo vindicatus*, important study of the foundations of geometry and precursor of non-Euclidean geometry.

A.D. 1733. Death of Jacob Hermann, early Swiss writer on the differential calculus.

A.D. 1833. Birth of Clebsch, Prussian algebraist and geometer.

A.D. 1833. Birth of L. Fuchs, German mathematician who made contributions to the study of linear differential equations.

A.D. 1833. Organization of the Statistical Section of the British Association for the Advancement of Science, due largely to the visit and influence of the Belgian statistician, Quetelet.

A.D. 1833. Death of the outstanding French analyst, Legendre, who greatly enriched mathematics by important contributions especially on elliptic integrals, theory of numbers, attraction of ellipsoids, and least squares.

A.D. 1833. Benjamin Peirce became professor of mathematics and natural philosophy at Harvard University, marking an epoch in the history of mathematics teaching at Harvard and in the United States. (See *School and Society*, 35: 533, April 16, 1932 and this MONTHLY, January 1925.)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 42. *Proposed by C. A. Rupp, Pennsylvania State College.*

If *abcde* and *baced* are squares, and $c+d$ and $b+e$ are successive primes, determine the five distinct digits a, b, c, d and e, and show that the solution is unique.

E 43. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

In the following simple multiplication the x's represent unknown digits to be determined. Show that there are just two solutions.

$$\begin{array}{rrrrr} & & x & x & x \\ & & & x & x \\ \hline & & x & x & x \\ & x & x & 4 & \\ \hline x & x & x & 1 & 7 \end{array}$$

E 44. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

ABC is an isosceles triangle with $AB=AC$. ADB is a right triangle with D,

the vertex of the right angle, on the opposite side of AB from C. Angle DAB is equal to angle BAC, and DF and CE are perpendicular to AB and AD at F and E respectively. Prove that AF and FB differ by AE.

E 45. *Proposed by W. R. Ransom, Tufts College.*

The ellipse of minimum area which can be circumscribed about a pair of equal, tangent circles, passes through the centers of its largest circles of curvature, and these centers and the two foci are the vertices of a square.

E 46. *Proposed by B. H. Brown, Dartmouth College.*

Show that $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+1/N$ is never an integer for any N.

E 47. *Proposed by D. C. Duncan, University of California at Berkeley.*

By methods of elementary plane geometry construct an equilateral triangle having a vertex upon each of three general lines in a plane, given the position of one vertex. Consider the case when the lines are parallel, and also the case in which the three lines are replaced by three concentric circumferences. What determines the number of solutions in the last case?

SOLUTIONS

E 18. [1933, 51] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Of all the right triangles whose areas exceed a million square units and whose three sides are integers without common factor, find that one whose perimeter is a minimum.

Solution by W. R. Ransom, Tufts College

When $a^2+b^2=c^2$, we have $a=2qr$, $b=r^2-q^2$, $c=r^2+q^2$, area $A=qr(r^2-q^2)$, and perimeter $P=2rq+2r^2$. Now $10^6<A$, and P is to be a minimum.

Since $2A=(rq-q^2)P$, $(rq-q^2)P<2{,}000{,}000$; and we shall try to get P a minimum by making $(rq-q^2)$ a maximum. This requires that $r=2q$, which gives $A=6q^2=10^6$ as an approximation, whence $q=20$, $r=40$, $P=4800$, and $A=960{,}000$ as values close to the desired solution.

We must now determine Δq and Δr so as to increase the area by at least 40,000, but increase the perimeter as little as possible. When $q=20$ and $r=40$ we find that

$$\partial A/\partial q = r^3 - 3q^2r = 16{,}000, \qquad \partial A/\partial r = 3r^2q - q^3 = 88{,}000,$$
$$\partial P/\partial q = 2r = 80, \qquad \partial P/\partial r = 2q + 4r = 200.$$

From these facts we set up the equations,

$$\Delta A = 16{,}000\Delta q + 88{,}000\Delta r = 40{,}000$$
$$\Delta P = 80\Delta q + 200\Delta r = 0$$

which give $\Delta q=-2.08$, $\Delta r=+.83$. Consequently we take $\Delta q=-2$, $\Delta r=1$, getting our new q and r as 18 and 41 respectively. As we change q by one either way from 18, or r by one either way from 41, or both, we get either a larger

perimeter or else too small an area. Hence $q=18$ and $r=41$ give the desired solution. Then the sides of the triangle are $a=1476=4\cdot 9\cdot 41$, $b=1356=23\cdot 59$, $c=2005=5\cdot 401$, and the perimeter is 4838, and the area is 1,001,466.

Solved also by J. Rosenbaum, E. E. Whitford, and the proposer.

E 20. [1933, 51]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The following letters represent the digits of a simple multiplication:

$$\begin{array}{cccccc} & & A & B & C \\ & & & B & D \\ \hline & C & E & E & B \\ F & B & C & D & \\ \hline F & G & C & G & B \end{array}$$

Solve and show that the solution is unique.

Solution by W. E. Buker, Leetsdale, Pa.

Since 1] $D\times C=$a number whose right hand digit is B,

2] $B\times C=$a number whose right hand digit is D,

3] neither B, C nor D could be 1,

4] $B+C<10$ (as no number is carried to give the F in the product).

Therefore the only possible values satisfying conditions [1] to [4] are $B=2$, $C=4$, $D=8$, as may readily be verified by actual trial.

From these facts it follows that $A=6$, $E=9$, $F=1$, $G=7$, and the actual multiplication was

$$\begin{array}{cccccc} & & 6 & 2 & 4 \\ & & & 2 & 8 \\ \hline & 4 & 9 & 9 & 2 \\ 1 & 2 & 4 & 8 & \\ \hline 1 & 7 & 4 & 7 & 2 \end{array}$$

Since all possibilities have been investigated, the solution is unique.

Solved also by Mannis Charosh, M. L. Constable, W. R. Ransom, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 21. [1933, 110]. *Proposed by V. F. Ivanoff, San Francisco.*

Prove that the differences of squares of two consecutive numbers in the third diagonal of the Pascal Triangle:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, \text{etc.}$$

is always a perfect cube.

Solution by L. S. Johnston, University of Detroit

The rth term of the third diagonal is ${}_{r+1}C_2$, and since ${}_{r+1}C_2=\frac{1}{2}r(r+1)$, we have ${}_{r+1}C_2=1+2+3+\cdots+r$.

Now it is well known that the sum of the cubes of the first r integers is equal to the square of the sum of those integers.

Consequently,

$$({}_{r+1}C_2)^2 - ({}_rC_2)^2 = (1 + 2 + 3 + \cdots + r)^2 - (1 + 2 + 3 + \cdots + [r - 1])^2 = r^3.$$

Solved also by Max Astrachan, S. F. Bibb, W. E. Buker, F. J. Feinler, D. W. Hall, C. A. Rupp, Simon Vatriquant, R. N. Walter and T. R. C.Wilson.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3616. *Proposed by J. B. Reynolds, Lehigh University.*

What is the surface of buoyancy for a homogeneous cube of edge a, if it floats with one face horizontal?

3617. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the center and the principal axes of an ellipse, and a line A anywhere in the plane of the ellipse, it is required by a ruler-compass construction and without drawing the ellipse to construct those normals to the ellipse that are parallel to the line A.

3618. *Proposed by N. A. Court, University of Oklahoma.*

The sum of the medians of a tetrahedron (i.e. the lines joining the vertices to the centroids of the opposite faces) is smaller than two thirds and greater than four ninths of the sum of the edges of the tetrahedron.

3619. *Proposed by V. F. Ivanoff, Berkeley, California.*

A plane cuts the edges OA, OB, OC of a parallelopiped in points A', B', C', respectively, and the diagonal OP in a point P'. Prove that

$$\frac{OA}{OA'} + \frac{OB}{OB'} + \frac{OC}{OC'} = \frac{OP}{OP'}.$$

3620. *Proposed by S. A. Corey, Des Moines, Iowa.*

Find solutions of the functional equation

$$a^4u^2 + u^2 + 2a^2u^2 + a^4 + 2a^2 = v^2$$

in which u and v are to be determined as rational functions of a.

3621. *Proposed by A. S. Levens, University of Minnesota.*

Extend the graphical method for the solution of real roots of a quadratic, as given in Dickson's *First Course in the theory of equations*, p. 29, to permit the reading of complex roots.

3622. *Proposed by Otto Dunkel, Washington University.*

A set of circles pass through a fixed point A and each circle of the set is tangent to a fixed circle I which does not contain A. Each circle of the set cuts the two tangents from A to I in a pair of corresponding points. Prove by synthetic geometry that the envelope of the straight lines joining the pairs of corresponding points is a circle.

This is the converse of the theorem in 3416 [1930, 157] an analytic proof of which is given 1930, 559.

3623. *Proposed by H. Grossman, New York City.*

Let A and B be two fixed points in a plane and P be a third point. Rotate AP from A as center through the fixed angle θ and then stretch it in the fixed ratio $k{:}1$ to position AP_A. Rotate and stretch BP similarly to position BP_B. The rotations may be both clockwise, both counterclockwise, or either one clockwise and the other one counterclockwise. Let P_0 be the midpoint of P_AP_B. If P describes any curve C in the plane, then the locus of P_0 is a curve C_0 similar to C. If the rotations are both in the same sense, the linear scale of C_0 to C is $k{:}1$; if the rotations are in opposite senses, the linear scale of C_0 to C is $k \cos\theta{:}1$.

This is a generalization of 3455 [1930, 477].

SOLUTIONS

3517. [1931, 539]. *Proposed by Morgan Ward, California Institute of Technology.*

Let $\Delta_N(a, \omega)$ denote the determinant

$$\begin{vmatrix}
\binom{1}{0} & 1-\omega & 0 & \cdots & 0 & 0 \\
\binom{2}{0} & \binom{2}{1} & 1-a\omega & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\cdot & \cdot & \cdot & \cdots & 1-a^{N-3}\omega & 0 \\
\cdot & \cdot & \cdot & \cdots & \binom{N-1}{N-2} & 1-a^{N-2}\omega \\
\binom{N}{0} & \binom{N}{1} & \binom{N}{2} & \cdots & \binom{N}{N-2} & \binom{N}{N-1}
\end{vmatrix}$$

Establish the formula

$$a^{N(N-1)/2}\Delta_N(a, a^{-N+1}) = \Delta_N(a, a)$$
$$= (1 + a)(1 + a + a^2) \cdots (1 + a + a^2 + \cdots + a^{N-1}).$$

Solution by the Proposer

Consider the system of N equations in the unknowns $x_1, \cdots, x_N$

$$\binom{r}{0} + \binom{r}{1}x_1 + \cdots + \binom{r}{r-1}x_{r-1} + (1 - a^r)x_r = 0, \qquad (r = 1, \cdots, N). \tag{1}$$

On solving for x_N by determinants, we find that

$$x_N = \frac{(-1)^N\Delta_N(a, a)}{(1 - a)(1 - a^2) \cdots (1 - a^N)}.$$

Hence

$$\Delta_N(a, a) = (-1)^N(1 - a)(1 - a^2) \cdots (1 - a^N)x_N. \tag{2}$$

Now by actual calculation, we find on solving the first equation of (1) for x_1, and the first two for x_2, that

$$x_1 = \frac{-1}{1 - a}, \qquad x_2 = \frac{1}{(1 - a)^2}.$$

If we assume that $x_k = (-1)^k(1-a)^{-k}$, $k = 1, 2, \cdots, r-1$, we have from the equations (1)

$$\binom{r}{0} - \binom{r}{1}\frac{1}{1 - a} + \binom{r}{2}\frac{1}{(1 - a)^2} - \cdots + (-1)^{r-1}\frac{1}{(1 - a)^{r-1}} + (1 - a^r)x_r = 0.$$

Hence

$$x_r = \frac{-1}{1 - a^r}\left\{\left(1 - \frac{1}{1 - a}\right)^r - \frac{(-1)^r}{(1 - a)^r}\right\} = \frac{(-1)^r}{(1 - a)^r}.$$

Thus by induction, $x_N = (-1)^N(1-a)^{-N}$. On substituting this value of x_N in (2), we obtain the second part of our formula; namely

$$\Delta_N(a, a) = (1 + a)(1 + a + a^2) \cdots (1 + a + a^2 + \cdots + a^{N-1}). \tag{3}$$

It does not seem possible to establish the first part of the formula in any such simple manner. Since both $a^{N(N-1)/2}\Delta_N(a, a^{-N+1})$ and

$$F_N(a) = (1 + a)(1 + a + a^2) \cdots (1 + a + a^2 + \cdots + a^{N-1})$$

are polynomials in a of degree $\frac{1}{2}N(N-1)$ and leading coefficients unity we shall have proved the first part of our formula if we can show that the determinant is

divisible by $F_N(a)$. To prove the latter result, it is sufficient to show that *every root of* $F_N(a)=0$ *is a root of* $a^{N(N-1)/2}\Delta_N(a, a^{-N+1})=0$ *of equal or greater multiplicity.*

Consider first the roots of $F_N(a)=0$. Since $a=1$ is not a root, every root is a primitive root of unity of degree $k \geqq 2$ and $\leqq N$. Let $a=\rho$ be such a root. Since $F_N(a)$ may be written $(1-a)^{-N+1}(1-a^2)(1-a^3)\cdots(1-a^N)$, and $\rho^n-1=0$ when and only when n is divisible by k, ρ is a root of $F_N(a)=0$ of multiplicity $r=[N/k]$, where $[N/k]$ denotes as usual the greatest integer in N/k.

There are in all $\phi(k)[N/k]$ such primitive roots of degree k, $\phi(k)$ being the totient of k. Since

$$\sum_{k=2}^{N}\phi(k)\left[\frac{N}{k}\right]=\tfrac{1}{2}N(N-1),$$

all the roots of $F_N(a)=0$ are thus accounted for. In particular, every primitive Nth root of unity is a root of $F_N(a)=0$ of multiplicity unity.

It only remains to show that $a=\rho$ *is a root of* $a^{N(N-1)/2}\Delta_N(a, a^{-N+1})=0$ *of multiplicity* $\geqq [N/k]$. For this step we need the following two lemmas which we shall establish later.

Lemma 1. *The determinant*

$$\begin{vmatrix}
\binom{M}{M-1} & 1-a^{M-1}\omega & 0 & \cdots & 0 \\
\binom{M+1}{M-1} & \binom{M+1}{M} & 1-a^{M}\omega & \cdots & \vdots \\
\binom{M+2}{M-1} & \binom{M+2}{M} & \binom{M+2}{M+1} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{M+N-1}{M-1} & \binom{M+N-1}{M} & \binom{M+N-1}{M+1} & \cdots & \binom{M+N-1}{M+N-2}
\end{vmatrix}$$

has the value

$$\binom{M+N-1}{M-1}\Delta_N(a, a^{M-1}\omega).$$

Lemma 2. *If* ρ *is a primitive* k*th root of unity, then*

$$\Delta_N(\rho, \rho^{-N+1}) = C\{\Delta_k(\rho, \rho)\}^r\Delta_{N-rk}(\rho, \rho^{-N+1}),$$

where C *is a rational integer, and* r *is the greatest integer in* N/k.

Our result now follows immediately from lemma 2. We have seen that $a=\rho$ is a root of multiplicity $[N/k]$ of $F_N(a, a)=0$. On setting $N=k$ in formula (3), we see that it is a root of multiplicity one of $\Delta_k(a, a)=0$. Hence by lemma 2, it is a root of multiplicity at least $[N/k]$ of $a^{N(N-1)/2}\Delta_N(a, a^{-N+1})=0$.

The proofs of the lemmas will be sufficiently illustrated by special cases. Take $M=3$, $N=4$ and let D be the resulting determinant in lemma 1. Then

$$D=\begin{vmatrix} \frac{3!}{2!1!} & 1-a^2\omega & 0 & 0 \\ \frac{4!}{2!2!} & \frac{4!}{3!1!} & 1-a^3\omega & 0 \\ \frac{5!}{2!3!} & \frac{5!}{3!2!} & \frac{5!}{4!1!} & 1-a^4\omega \\ \frac{6!}{2!4!} & \frac{6!}{3!3!} & \frac{6!}{4!2!} & \frac{6!}{5!1!} \end{vmatrix},$$

$$=\frac{6!}{2!}\begin{vmatrix} \frac{1}{1!} & 1-a^2\omega & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1-a^3\omega & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1-a^4\omega \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

on performing upon the first determinant the operations

$$\frac{2!}{6!}\text{ col }1,\ \frac{3!}{6!}\text{ col }2,\ \frac{4!}{6!}\text{ col }3,\ \frac{5!}{6!}\text{ col }4;\ \frac{6!}{3!}\text{ row }1,\ \frac{6!}{4!}\text{ row }2,\ \frac{6!}{5!}\text{ row }3,\ \frac{6!}{6!}\text{ row }4.$$

Finally, performing upon the second determinant the operations

$$\frac{1}{1!}\text{ col }2,\ \frac{1}{2!}\text{ col }3,\ \frac{1}{3!}\text{ col }4;\ 2!\text{ row }2,\ 3!\text{ row }3,\ 4!\text{ row }4,$$

we find that

$$D=\binom{6}{2}\Delta_4(a, a^2\omega).$$

For lemma 2, take $N=8$, and let $a=\rho$ be a primitive cube root of unity, so that $k=3$, $r=[N/k]=2$, $N-rk=2$. Then on replacing ρ^3 by 1 whenever it appears, we see that

$$\Delta_8(\rho, \rho^{-7}) = \Delta_8(\rho, \rho^2)=$$

$$\begin{vmatrix}
\binom{1}{0} & 1-\rho^2 & 0 & & & & & \\
\binom{2}{0} & \binom{2}{1} & 0 & 0 & 0 & & & \\
 & \binom{3}{1} & \binom{3}{2} & 1-\rho & 0 & 0 & & \\
 & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & 1-\rho^2 & 0 & & \\
 & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & 0 & & \\
 & & & & \binom{6}{4} & \binom{6}{5} & 1-\rho & 0 \\
 & & & & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & 1-\rho^2 \\
\binom{8}{0} & \binom{8}{1} & \binom{8}{2} & \binom{8}{3} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{8}{7}
\end{vmatrix}$$

It is evident that this last determinant is the product of the three determinants set off by dotted lines. On applying lemma 1 to the last two of these, we see that

$$\Delta_8(\rho, \rho^{-7}) = \binom{8}{5}\binom{5}{2}\{\Delta_3(\rho, \rho)\}^2\Delta_2(\rho, \rho^{-7}).$$

A Note by Otto Dunkel. The determinant $\Delta_N(a, \omega)$ of order N is of the type in which the elements are of the form

$$(i, j) = \binom{i}{j-1}, \quad 1 \leqq i \leqq N,\ 1 \leqq j \leqq i,$$

$$(i, i+1) = a_{i,i+1}, \quad 1 \leqq i \leqq N-1,$$

$$(i, j) = 0, \quad 1 \leqq i \leqq N-2,\ i+2 \leqq j \leqq N,$$

where i, j denote the row and column, respectively, of the element, and where the elements $a_{i,i+1}$ are arbitrary. These latter elements lie in a line parallel to the principal diagonal and just above it; call this line the a parallel for brevity. Such a determinant has the property that its value is unaltered if the elements in the a parallel are written in the reverse order; such a reversal may be obtained by rotating the a parallel through 180° about the secondary diagonal as an axis while each other element remains fixed. This may be proved as follows: Multiply the elements in the j column by $(N-j+1)!(j-1)!$, and divide the elements in the i row by $(N-i)!i!$. When this is done for all the columns and

rows, it will be easily seen that the elements $a_{i,i+1}$ are unaltered, whereas any other element (i, j) not zero is replaced by the element which was at $(N-j+1, N-i+1)$. In other words the elements, except those in the a parallel, have been interchanged by a rotation through 180° about the secondary diagonal. The value of the determinant has not been changed, since the multiplications and divisions cancel in the end. Now rotate all the elements of the resulting determinant through 180° about the secondary diagonal. This operation does not alter the value of the determinant, and the result is that each element not in the a parallel has been restored to its original position, but the succession of the elements in the a parallel has been reversed in order. This completes the proof of the proposition.

Now consider a determinant D of this kind in which $a_{i,i+1}=1-x^i$. To each element of the second column add the product of $x-1$ times the corresponding element of the first column; to the elements of the third column add the products of $x-1$ times the resulting elements of the second column; and continue in this same manner. In the end we have for the element (i, i) in the ith row

$$(i, i) = \sum_{k=1}^{i} \binom{i}{i-k} (x-1)^{k-1},$$

$$= \frac{1}{x-1}\left\{ \sum_{k=0}^{i} \binom{i}{k} (x-1)^k - 1 \right\} = \frac{x^i - 1}{x-1}.$$

In the same row we have zero as the result for $(i, i+1)$; and hence zeros for all following elements. Thus

$$D = \prod_{i=1}^{N} (1 + x + x^2 + \cdots + x^{i-1}).$$

Next consider D', where the element $a_{i,i+1}=1-y^{N-i}$, $y=x^{-1}$. As shown above D' is unaltered if each $a_{i,i+1}$ is replaced by $a_{N-i,N-i+1}$, or if $1-y^{N-i}$ is replaced by $1-y^i$. When this is done D' is evaluated by replacing x in D by y. Thus we have finally the desired result

$$x^{N(N-1)/2} D' = D = \prod_{i=1}^{N} (1 + x + x^2 + \cdots + x^{i-1}).$$

3561. [1932, 359]. *Proposed by Emmanuel Wad, Baltimore, Md.*

The number 12345678 is not divisible by 11, but by placing the eight figures in different orders we can form other numbers which are divisible by 11. Determine how many such numbers can be formed.

Solution by D. B. Perry, Stanford University and J. D. Hill, Brown University.

We first note that

$$N = a_0 10^n + a_1 10^{n-1} + \cdots + a_n \equiv 0 \bmod 11,$$

if and only if

$$D = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n \equiv 0 \bmod 11.$$

This, of course, follows merely from the fact that $N+D\equiv 0 \bmod 11$, if n is odd, and that $N-D\equiv 0 \bmod 11$, if n is even. In the present example, $n=7$, and the a's form a permutation of 1, 2, $\cdots$, 8. Then if we place $a_0+a_2+a_4+a_6=d$, we have $a_1+a_3+a_5+a_7=36-d$. The condition that N be a multiple of 11 is

$$D = 2(d - 18) \equiv 0 \bmod 11,$$

which requires that $d=7$, 18, or 29. Of these values, however, 7 and 29 are evidently impossible; hence $d=36-d=18$.

It is not difficult to verify that the given digits admit eight distinct sets of four numbers each whose sum is 18. These are $(1, 2, 7, 8)_1$, $(1, 3, 6, 8)_2$, $(1, 4, 5, 8)_3$, $(1, 4, 6, 7)_4$ $(2, 3, 5, 8)_4$, $(2, 3, 6, 7)_3$, $(2, 4, 5, 7)_2$, $(3, 4, 5, 6)_1$, where complementary sets have the same subscript. We may then choose the terms of d in 8 ways, and corresponding to each such choice the terms of $36-d$ are uniquely determined. But each set of four digits may be permuted independently in 24 ways. Hence, there may be formed with the given digits $8\cdot 24\cdot 24=4608$ different multiples of 11. The smallest and largest of these are, respectively, 12346587 and 87653412.

Solved also by H. T. R. Aude, M. G. Boyce, W. E. Buker, H. A. Campbell, Leverett Davis, Jr., G. H. Graves, H. Grossman, David Katz, Theodore Lindquist, L. C. Mathewson, Ruth G. Mason, A. Pelletier, B. D. Roberts, and F. Underwood.

3562. [1932, 429]. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

If $\phi(x, y, z)=0, \psi(x, y, z)=0$, prove that

$$\begin{vmatrix} d^2x & d^2y & d^2z \\ \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = 0,$$

under suitable conditions.

Solution by J. M. Feld, Brooklyn College.

Let the surfaces $\phi=0$ and $\psi=0$ determine a curve C and let s be the distance measured on C from a fixed point on the curve. Then dx/ds, dy/ds and dz/ds are the direction cosines of a tangent to C, and

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

Differentiating with respect to s we obtain

$$\frac{d^2x}{ds^2}\frac{dx}{ds} + \frac{d^2y}{ds^2}\frac{dy}{ds} + \frac{d^2z}{ds^2}\frac{dz}{ds} = 0. \tag{1}$$

In addition we have

$$\phi_x \frac{dx}{ds} + \phi_y \frac{dy}{ds} + \phi_z \frac{dz}{ds} = 0 \tag{2}$$

$$\psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} + \psi_z \frac{dz}{ds} = 0. \tag{3}$$

Eliminating dx/ds, dy/ds, and dz/ds and multiplying by ds^2, we obtain the stated relationship.

Solved also by J. D. Leith.

A Note by Otto Dunkel. Consider as the independent parameter the time t at which a point is at x, y, z of the curve C and let s denote the length of arc to this point as in the solution above. Denote by A, B, C the cofactors of the first row of the determinant in the problem, and set $R=(A^2+B^2+C^2)^{1/2}\neq 0$, $s''=d^2s/dt^2$. Then the given determinant may be written

$$Ad^2x + Bd^2y + Cd^2z = Rs''(dt)^2.$$

This easily follows by resolving the components of the acceleration, x'', y'', z'', along the path s, which has direction components A/R, B/R, C/R; the sum of the resolved parts must be s''.

The determinant is zero if, and only if, $s''=0$; or if $s=at+b$, where a and b are constants, $a\neq 0$. Hence the suitable conditions are that we must choose our parameter t as a linear integral function of s.

3564. [1932, 429]. *Proposed by H. T. R. Aude, Colgate University.*

A determinant of order $n+1$ has for the elements of its first and second rows, respectively, the successive powers of α and x with exponents from 0 to n inclusive. The third row is the derivative of the second row, and the elements of any row after the second are given by the relation

$$\alpha_{i+1,j} = \frac{1}{i-1}\frac{d\alpha_{i,j}}{dx}, \quad i = 2, 3, \cdots, n.$$

Prove that this determinant has the value $(x-\alpha)^n$.

Solution by R. E. Moritz, University of Washington

This problem is an easy corollary to the following general theorem: Let

$$F(x) = \begin{vmatrix} f_0(x), & f_1(x), & \cdots, f_n(x) \\ f_0(\alpha), & f_1(\alpha), & \cdots, f_n(\alpha) \\ f_0'(\alpha), & f_1'(\alpha), & \cdots, f_n'(\alpha) \\ f_0''(\alpha), & f_1''(\alpha), & \cdots, f_n''(\alpha) \\ \cdots & \cdots & \cdots \\ f_0^{(n-1)}(\alpha), & f_1^{(n-1)}(\alpha), & \cdots, f_n^{(n-1)}(\alpha) \end{vmatrix},$$

where $f_i(x)$, $i=0, 1, \cdots, n$, are $n+1$ distinct polynomials in x and $f_i^{(k)}(x)$ is the kth derivative of $f_i(x)$. If m, the highest degree of any one of these polynomials, is not less than n, then $F(x)$ contains $(x-\alpha)^n$ as a factor.

Proof. It is evident that $F(x)$ is a polynomial of degree m in x whose kth derivative is

$$F^{(k)}(x) = \begin{vmatrix} f_0^{(k)}(x), & f_1^{(k)}(x), & \cdots, f_n^{(k)}(x) \\ f_0(\alpha), & f_1(\alpha) & , \cdots, f_n(\alpha) \\ f_0'(\alpha), & f_1'(\alpha), & \cdots, f_n'(\alpha) \\ f_0''(\alpha), & f_1''(\alpha), & \cdots, f_n''(\alpha) \\ \cdots & \cdots & \cdots \\ f_0^{(n-1)}(\alpha), & f_1^{(n-1)}(\alpha), & \cdots, f_n^{(n-1)}(\alpha) \end{vmatrix}.$$

Now obviously both $F(x)$ and $F^{(k)}(x)$ vanish when $x=\alpha$, provided k is less than n. It therefore follows from the well-known theorem regarding multiple roots that $F(x)$ contains $(x-\alpha)^n$ as a factor.

Cor. 1. If $m=n$, $F(x)=c(x-\alpha)^n$, where c is a constant which may be determined by comparing coefficients of like powers of x of the two members of the equation.

Cor. 2. If $f_i(x)=x^i$, then $c=1!2!3!4!\cdots(n-1)!(-1)^n$.

Cor. 3. If $f_i(x)=x^i$ and the coefficients of the rows after the second are those specified in the problem, then $c=(-1)^n$.

Cor. 4. If in Cor. 3, we interchange x and α, we have the determinant, as specified in the problem as originally stated, equal to $(-1)^n(\alpha-x)^n=(x-\alpha)^n$.

Solved also by Frank Ayres, Jr., G. A. Baker, Joseph Lev, W. V. Parker, S. S. Shu, F. W. Sparks, H. S. Thurston, F. Underwood, and the proposer.

3565. [1932, 430]. *Proposed by Orrin Frink, Jr., Pennsylvania State College.* Find the ellipse of least area circumscribing a given triangle.

Solution by F. Underwood, University College, Nottingham, England

Let PQR be the given triangle in a plane α. We can project orthogonally upon another plane α' in such a way that the corresponding points $P'Q'R'$ will be the vertices of an equilateral triangle. (See solution of problem 2895 [1933, 274]; or Müller's *Darstellende Geometrie*, vol. I, p. 97). Let C' be the circle circumscribed to $P'Q'R'$, and let E be the corresponding ellipse circumscribed to PQR in the plane α. The ratio of the area of the circle C' to the inscribed equilateral triangle $P'Q'R'$ is $r=4\pi/3\sqrt{3}$; and, the equilateral triangle being the maximum triangle that can be inscribed in a given circle, the ratio of the area of any circle to that of any triangle inscribed in it can never be less than r. Then since orthogonal projection leaves ratios of areas unchanged, and since any ellipse can be projected orthogonally into a circle, it follows that the ratio of the area of *any* ellipse to that of *any* triangle inscribed in it can never be less

than r. But it also follows that the ratio of the area of the ellipse E to that of the triangle PQR is r, and hence E must be the required ellipse of minimum area.

To construct E in the plane α, it is only necessary to observe that an orthogonal projection preserves parallelism, and hence E must be the ellipse circumscribing the triangle PQR and tangent at each vertex to a line parallel to the opposite side. This ellipse can be constructed by well-known methods, such as are given in Minchin and Dale's *Mathematical Drawing*.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Dr. Henry Eyring, research associate in chemistry at Princeton University, has been awarded the ninth annual $1,000 prize of the American Association for the Advancement of Science, for his paper entitled *Quantum mechanics and chemistry with particular reference to reactions involving conjugate double bonds*, presented at the Atlantic City meeting.

Dr. Charles Roos formerly assistant professor of mathematics at Cornell University and later permanent secretary for the American Association for the Advancement of Science has been awarded a Guggenheim fellowship. He plans to write a book on dynamical economics in consultation with European scholars.

The Alfred Noble prize for the best technical paper of the year on a topic in engineering has been awarded to Frank M. Starr, of the General Electric Company, for his paper entitled *Equivalent Circuits, II*.

Professor H. A. Simmons, of Northwestern University, on sabbatical leave this semester, spent two months in study at Princeton and will be abroad until fall in France, Germany and Italy.

On February 3, 1933, at Tulsa, the teachers of mathematics in the colleges of Oklahoma organized an Oklahoma Section of the Mathematical Association of America, electing the following officers: Chairman, Professor N. A. Court, University of Oklahoma; Vice Chairman, Professor A. M. Wallace, East Central Teachers College; Secretary, Professor E. F. Allen, Oklahoma A. and M. College. By-laws were adopted and submitted to the Trustees of the Association.

The following seventy-eight doctorates with mathematics or mathematical physics as major subject were conferred during 1932 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

J. R. Abernethy, Michigan, On the application of divided differences to approximation.

Alice L. Ambrose, Wisconsin, June, major in philosophy, In defense of an extensional logic.

Sister Mary Nicholas Arnoldy, Catholic, October, minors in physics and education, The reality of the double tangents of the rational symmetric quartic curve.

N. H. Ball, Massachusetts Institute of Technology, June, minor in physics, Projective geometry of element-manifolds.

R. A. Beaver, Cornell, September, minor in physics, Finite plane euclidean geometry.

A. H. Black, Cornell, June, minor in physics, Types of involutorial space transformations associated with certain rational curves.

J. J. Brady, California (Berkeley), May, major in physics, A quantitative study of the photo-electric properties of thin alkali metal films.

G. S. Bruton, Wisconsin, June, Certain aspects of the theory of equations for a pair of matrices.

R. C. Bullock, Chicago, June, Non-conjugate osculating quadrics of a curve on a surface.

J. H. Butchart, Illinois, June, minor in physics, Helices in euclidean n-spaces.

R. H. Cameron, Cornell, September, minor in physics, Almost periodic transformations.

J. F. Carlson, California (Berkeley), May, major in physics, The energy losses of fast particles.

J. M. Clarkson, Cornell, June, minor in physics, Some involutorial line transformations interpreted as points of V of S.

H. H. Conwell, Wisconsin, June, Linear associative algebras of infinite rank whose elements satisfy finite algebraic equations.

Frances T. Cope, Radcliffe, June, Formal solutions of irregular linear differential equations.

A. H. Diamond, California (Berkeley), December, The complete existential theory of the Whitehead-Huntington set of postulates for the algebra of logic.

J. L. Doob, Harvard, June, The boundary values of analytic functions.

C. H. Fischer, Iowa, June, minor in economics, On correlation surfaces of sums with a certain number of random elements in common.

D. G. Fulton, Michigan, Generalizations of the Cauchy integral formula.

C. E. M. Gewertz, Massachusetts Institute of Technology, June, major in electrical engineering, Synthesis of a finite four-terminal network whose driving point and transfer functions are prescribed.

G. D. Gore, Chicago, August, Surfaces associated with a congruence.

H. E. H. Greenleaf, Indiana, October, minor in physics, Curve approximation by means of functions analogous to the Hermite polynomials.

Harvey Hall, California (Berkeley), May, major in physics, Relativistic theory of the photo-electric effect.

C. H. Harry, Johns Hopkins, June, Concerning spaces without local cut points and a geometry of acyclic spaces.

M. C. Hartley, Illinois, June, minor in astronomy, Properties of algebraic, plane quintics which are invariant under finite collineation groups.

G. G. Havens, Wisconsin, June, major in physics, The magnetic susceptibility of some common gases.

M. R. Hestenes, Chicago, June, Sufficient conditions for the general problem of Mayer with variable end-points.

E. H. C. Hildebrandt, Michigan, Systems of polynomials connected with the Charlier expansions and the Pearson differential and difference equations.

Banesh Hoffmann, Princeton, April, On the spherically symmetric field in relativity.

I. O. Horsfall, Cornell, June, minor in physics, Transformations associated with the lines of a cubic, quadratic, or linear complex.

Kuen-Sen Hu, Chicago, June, The problem of Bolza and its accessory boundary value problem.

Ralph Hull, Chicago, August, The numbers of solutions of congruences involving only kth powers.

R. E. Huston, Chicago, August, Asymptotic generalizations of Waring's theorem.

R. D. James, Chicago, June, Analytical investigations in Waring's theorem.

M. W. Keller, Indiana, June, minor in physics, A study of the angular velocity about a point between the foci in Keplerian elliptic motion.

S. H. Kimball, Harvard, June, On rigid motions in four dimensions, with applications to the Laguerre geometry of three dimensions.

R. W. P. King, Wisconsin, June, major in electrodynamics, Characteristics of vacuum tube circuits having distributed constants at ultra-radio frequencies.

E. C. Klipple, Texas, June, Spaces in which there exist contiguous points.

H. L. Krall, Brown, June, An asymptotic expression of the characteristic values of the elliptic partial differential equations.

J. H. Kusner, Pennsylvania, June, On continuous curves with cyclic connection of higher order.

A. J. Lewis, Colorado, August, Solution of algebraic equations with one unknown by infinite series.

D. C. Lewis, Harvard, June, Infinite systems of ordinary differential equations with applications to certain second order non-linear partial differential equations of hyperbolic type.

C. F. Luther, Stanford, June, minor in physics, Concerning primitive groups of class U.

M. H. Martin, Johns Hopkins, June, On infinite orthogonal matrices.

R. S. Martin, California Institute of Technology, June, minor in physics, Contribution to the theory of functionals.

Ruth G. Mason, Chicago, March, Studies in the Waring problem.

David Moskovitz, Brown, June, Certain irregular non-homogeneous linear difference equations.

A. F. Moursund, Brown, October, On a method of summation of Fourier series.

S. B. Myers, Harvard, June, Sufficient conditions in the problem of the calculus of variations in n-space in parametric form and under general end conditions.

Leo Nedelsky, California (Berkeley), May, major in physics, Radiation from slow electrons.

Emma J. Olson, Chicago, August, Conjugate systems characterized by special properties of their ray congruences.

G. B. Price, Harvard, June, On the double pendulum and similar dynamical systems.

Irene Price, Indiana, October, minor in astronomy, On a certain type of polynomials.

E. J. Purcell, Cornell, June, Involutorial space Cremona transformations determined by non-linear null reciprocities.

A. W. Raab, Chicago, August, Jacobi's condition for multiple integral problems of the calculus of variations.

Francis Regan, Michigan, The application of the theory of admissible numbers to time series.

C. E. Rhodes, Cincinnati, June, Concerning the double Poisson integral and its derivatives.

Nathan Rosen, Massachusetts Institute of Technology, June, major in physics, Calculation of energies of diatomic molecules.

Helen G. Russell, Radcliffe, June, On the degree of convergence and overconvergence of polynomials of best simultaneous approximation to several functions analytic in distinct regions.

Jacob Sherman, Pennsylvania, June, On the numerators of the convergents of the Stieltjes continued fractions.

D. T. Sigley, Illinois, June, minor in physics, Groups involving a small number of complete sets of conjugates.

A. Sisk, Cornell, June, The plane symmetric quintic Cremona involutions.

A. H. Smith, Brown, October, On the summability of derived series of the Fourier-Lebesgue type.

A. E. Staniland, Pittsburgh, June, On the Segre curved four-space representation of the plane of two complex variables.

Mildred M. Sullivan, Radcliffe, June, On the derivatives of Newtonian and logarithmic potentials near the acting masses.

J. F. Thomson, Michigan, Motion of the electrons of the helium atom.

Sister Mary Domitilla Thuener, Catholic, June, minors in physics and education, On the number and reality of the self-symmetric quadrilaterals in- and circumscribed to the triangular-symmetric rational quartic.

E. W. Titt, Princeton, June, Systems of partial differential equations and their characteristic surfaces.

A. W. Tucker, Princeton, June, An abstract approach to manifolds.

C. H. Vehse, Brown, June, Acceleration stresses in a heavy wire rope.

C. W. Vickery, Texas, June, minor in philosophy, Spaces in which there exist uncountable convergent sequences of points.

J. P. Vinti, Massachusetts Institute of Technology, June, major in physics, The variational calculation of atomic wave functions.

E. H. Wagner, Michigan, A treatment of systems of linear difference equations having faculty series as coefficients.

A. M. Wahl, Pittsburgh, June, minor in mechanics. Bending of semi-circular plates with and without radially slotted peripheral portions.

A. E. Whitford, Wisconsin, June, major in physics, Zeeman effect of the K II spectrum.

Hassler Whitney, Harvard, February, The coloring of graphs.

Kamcheung Woo, California (Berkeley), May, Projective-transformation group on a hyperquadric in four-dimensional space.

Marguerite L. Zeigel, Missouri, August, minor in physics, Some invariant properties of a two-dimensional surface in hyperspace.

Dr. H. M. Bacon, of Stanford University, California, would like to secure one or more copies of a book by H. S. Carslaw, *The Elements of Non-Euclidean Plane Geometry and Trigonometry* (Longmans, Green and Company, 1916) for use in connection with a course which he is giving. The book has recently gone out of print. Anyone having a copy for sale is requested to communicate with Dr. Bacon.

THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Southern Methodist University in Dallas, Texas, on Saturday, February 11th, 1933. Those attending the meeting numbered thirty-one at the morning session, and ninety-seven at the afternoon session; among these were the following fourteen members of the Association: L. W. Blau, L. M. Blumenthal, H. E. Bray, J. E. Burnam, Alice C. Dean, Nat Edmonson, G. C. Evans, L. R. Ford, C. M. Howard, E. H. Jones, P. K. Rees, W. A. Rees, B. P. Reinsch, C. R. Sherer.

The proceedings took place in the Hyer Physics Laboratory of Southern Methodist University. Lunch was served in the Virginia Dining Hall of the University. In the evening the members of the Texas Section and their guests met at a banquet served in Virginia Hall. The principal speaker at this dinner was Professor J. M. Bledsoe of the East Texas State Teachers' College, who gave by invitation a paper entitled "The outlook of mathematics in secondary education." Professor Bledsoe reached the conclusion that secondary school authorities are steadily tending to minimize the importance of mathematics in the secondary school curriculum in Texas. The quantity of mathematics required is becoming less and less, while the quality of the teaching deteriorates. The result is that the college mathematics staffs of the state are facing a problem which is becoming more severe.

The Texas Section of the Mathematical Association of America is attempting, as part of its policy, to establish greater contact with the teachers of mathematics in the secondary schools of Texas. As a result of special efforts in this direction a large number of such teachers attended this meeting.

The following papers were presented:

1. "Some applications of mathematics in geophysics" by Dr. L. W. Blau, Director of Geophysical Research, Humble Oil and Refining Company.
2. "A geometric study of the equation

$$dy/dx = (l_1x + m_1y)/(l_2x + m_2y)$$

by Professor L. R. Ford, The Rice Institute.

3. "Certain criteria in the interpretation of statistical coefficients" by Professor Lonnie Langston, Texas Technological College, by invitation.
4. "A determinantal characterization of d-cyclic and pseudo d-cyclic sets of points" by G. A. Garrett, The Rice Institute, by invitation.
5. "Psychological principles applied to the learning process and to testing in mathematics" by Professor B. P. Reinsch, Southern Methodist University.
6. "Does the present high school curriculum in mathematics prepare students for college?" by Professor C. R. Sherer, Texas Christian University.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles: